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The vortical flow above the drain-hole in a rotating vessel

By T. S. LUNDGREN
Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, Minnesota 55455

The problem of slow draining of incompressible fluid from a rotating cylindrical vessel is considered. Transient effects, internal viscosity, boundary friction and finite cylindrical boundaries are included. In addition a tornado model formulated by Turner is solved and compared with some experiments on rotating turbulence of Hopfinger, Browand & Gagne.

1. Introduction

While there is considerable literature on tornado-like flows, for instance the recent conference proceedings edited by Bengtsson & Lighthill (1982) and the review article by Maxworthy (1982) contained in that volume, the literature which is specifically related to the slow withdrawal of an incompressible fluid from a rotating container is more limited (Einstein & Li 1951; Rott 1958; Lewellen 1962; Andrade 1963). A number of aspects of this problem have not been fully treated and the transient problem appears not to have been considered at all. Some features of the steady problem can be better understood by approaching it through a transient analysis.

This work is partly motivated by a recent paper by Hopfinger, Browand & Gagne (1982) in which an oscillating grid in the bottom of a rotating liquid-filled cylinder produces a layer of turbulence near the bottom of the cylinder from which a large number (about 20) of intense vortices extend vertically to the top of the cylinder. These vortices can have a local vorticity as large as fifty times the background vorticity of the tank and disappear after a lifetime of about 20 rotation periods by catastrophic events, to be replaced by apparently new vortices. A plausible source for these vortices might be the existence of fairly persistent suction sites within the turbulent layer caused by low subharmonics of the basic driving oscillatory motion. If such sites exist they might be expected to produce vortices similar to those produced by a rotating drain, by concentrating the vorticity of the rotating fluid as it is drawn into the site.

In the basic problem to be considered in this paper, liquid is contained between two infinite flat plates separated by distance H. Initially the plates and the liquid are rotating with angular velocity Ω, when a hole of radius R is opened on the axis of rotation of the bottom plate from which fluid is slowly withdrawn with average velocity $u(t/t_s)$. This function slowly increases to a final steady value u_∞ in a time of order t_s. The velocity profile at the hole is usually taken to be uniform, for simplicity, but could be arbitrarily specified in principle. In a real gravitational drain it would be determined by the details of the external plumbing.

A number of variations of the basic problem are also considered. The effects of a sidewall, internal viscosity and boundary friction are included. In an important modification, which will be called Turner's problem (Turner 1966), the prescribed velocity profile has a central downflow with a surrounding annular upflow such that there is no net flow from the tank. It is thought that the resulting tornado-like flow is close to what is seen in the Hopfinger, Browand and Gagne (hereinafter referred to as HBG) experiment.

The equations describing such flows are:

$$\frac{1}{r}\frac{\partial r u_r}{\partial r}+\frac{\partial u_z}{\partial z}=0; \tag{1.1}$$

$$\left[\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+u_z\frac{\partial}{\partial z}\right] r u_\theta=0; \tag{1.2}$$

$$\left[\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+u_z\frac{\partial}{\partial z}\right] u_r-\frac{u_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}; \tag{1.3}$$

$$\left[\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+u_z\frac{\partial}{\partial z}\right] u_z=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g. \tag{1.4}$$

Axial symmetry has been assumed and the neglected viscous terms will be included when needed. These equations may be made dimensionless by introducing new variables;

$$\left.\begin{aligned}\eta=\frac{r}{R};\quad \zeta=\frac{z}{H};\quad T=\Omega t;\quad U_z=\frac{u_z}{u_\infty};\quad U_r=\frac{Hu_r}{Ru_\infty};\\ U_\theta=\frac{u_\theta}{R\Omega};\quad P=\frac{p}{\rho R^2\Omega^2};\quad \epsilon=\frac{u_\infty}{H\Omega};\quad G=\frac{gH}{R^2\Omega^2};\end{aligned}\right\} \tag{1.5}$$

where ϵ is a Rossby number which will be assumed small. The equations are thus:

$$\frac{1}{\eta}\frac{\partial \eta U_r}{\partial \eta}+\frac{\partial U_z}{\partial \zeta}=0; \tag{1.6}$$

$$\left[\frac{\partial}{\partial T}+\epsilon\left[U_r\frac{\partial}{\partial \eta}+U_z\frac{\partial}{\partial \zeta}\right]\right]\eta U_\theta=0; \tag{1.7}$$

$$\epsilon\left[\frac{\partial}{\partial T}+\epsilon\left[U_r\frac{\partial}{\partial \eta}+U_z\frac{\partial}{\partial \zeta}\right]\right]U_r-\frac{U_\theta^2}{\eta}=-\frac{\partial P}{\partial \eta}; \tag{1.8}$$

$$\epsilon\left[\frac{H}{R}\right]^2\left[\frac{\partial}{\partial T}+\epsilon\left[U_r\frac{\partial}{\partial \eta}+U_z\frac{\partial}{\partial \zeta}\right]\right]U_z=-\frac{\partial P}{\partial \zeta}-G. \tag{1.9}$$

The boundary and initial conditions for the inviscid problem are:

$$U_z=0 \quad \text{at } \zeta=1; \tag{1.10}$$

$$U_z=F(T/T_s)\, W_B(\eta) \quad \text{at } \zeta=0; \tag{1.11}$$

where $F(T/T_s)=u(T/T_s)/u_\infty$, $T_s=\Omega t_s$ and

$$U_z=0,\quad U_r=0,\quad U_\theta=\eta \quad \text{at } T=0. \tag{1.12}$$

The function $W_B(\eta)$ is the velocity profile at the hole, taken to be zero when $\eta>1$; for uniform flow $W_B(\eta)=-1$ when $\eta<1$. The function $F(T/T_s)$ is the starting time dependence; $F(T/T_s)\to 1$ as $T/T_s\to\infty$.

The plan of the paper is as follows. In §2 the linearized version of the above problem is treated in order to set the stage and initiate the nonlinear problem. Two things are found; inertial oscillations associated with starting the drain may be suppressed by making T_s large (i.e. a gradual start); and, more importantly, the linearized problem is not uniformly valid at large time (U'_θ continues to increase with time). In §3 the nonlinear version of this problem is solved and a similar result is found; the tangential velocity continues to increase. This does not violate the equations but suggests that viscosity is important. In §4 internal viscosity is included and, while this modifies the swirling-velocity profile, the velocity continues to accelerate. In §5 boundary friction is included by using an approximate nonlinear version of the Ekman compatibility boundary condition introduced by Turner (1966). When there is friction on the bottom (the suction side) with a free upper surface, the swirling velocity still increases with time (this would of course finally cause a deep whirlpool depression in the free surface and violate the approximation of a flat surface). Only when friction on the top surface is included does this problem have an ultimate steady solution.

The results of §4 would appear to disagree with those of Einstein & Li (1951). They found a steady solution with internal friction but without top or bottom friction. Their problem differs from those described above in one boundary condition; the tangential velocity is maintained at a fixed value at a finite distance from the axis. In their experiments this is accomplished by inlets and baffles. This problem can be solved by the same methods as the others; the resulting solution tends to a steady state at large time when internal viscosity is included. The Einstein–Li boundary condition is discussed in §§3, 4 and 5. The shape of the free surface is also determined for the Einstein–Li problem in §4. In all other free-surface problems it is assumed that gravity is effectively large enough to make the surface flat.

In §6 Turner's problem is solved and the application to the HBG experiment is discussed.

2. The linearized problem

In the linearized version of the problem set by (1.6)–(1.12) a solution is sought in the form of a small perturbation from solid body rotation;

$$U_z = U_{z0} + \epsilon U_{z1} + \dots, \tag{2.1}$$

$$U_r = U_{r0} + \epsilon U_{r1} + \dots, \tag{2.2}$$

$$U_\theta = \eta + \epsilon U_{\theta 1} + \dots, \tag{2.3}$$

$$P = P_0 + \epsilon P_1 + \dots. \tag{2.4}$$

The lowest-order equations are thus

$$\frac{1}{\eta}\frac{\partial \eta U_{r0}}{\partial \eta} + \frac{\partial U_{z0}}{\partial \zeta} = 0, \tag{2.5}$$

$$\frac{\partial U_{\theta 1}}{\partial T} + 2U_{r0} = 0, \tag{2.6}$$

$$-\eta = -\frac{\partial P_0}{\partial \eta}, \tag{2.7}$$

$$0 = -\frac{\partial P_0}{\partial \zeta} - G. \tag{2.8}$$

To these must be added the first-order equations

$$\frac{\partial U_{r0}}{\partial T}-2U_{\theta 1}=-\frac{\partial P_1}{\partial \eta}, \tag{2.9}$$

$$\left[\frac{H}{R}\right]^2\frac{\partial U_{z0}}{\partial T}=-\frac{\partial P_1}{\partial \zeta}. \tag{2.10}$$

Eliminating P_1 from these last two equations gives

$$\frac{\partial^2 U_{r0}}{\partial \zeta\,\partial T}-2\,\frac{\partial U_{\theta 1}}{\partial \zeta}=\left[\frac{H}{R}\right]^2\frac{\partial^2 U_{z0}}{\partial \eta\,\partial T}. \tag{2.11}$$

Upon operating on (2.11) with $\partial^2\eta(\;)/\eta\,\partial T\,\partial\eta$ and using (2.5) and (2.6) to eliminate U_{r0} and $U_{\theta 1}$ one obtains a single equation for U_{z0}, namely

$$\frac{\partial^2}{\partial T^2}\left[\frac{\partial^2 U_{z0}}{\partial \zeta^2}+\left[\frac{H}{R}\right]^2\frac{1}{\eta}\frac{\partial}{\partial \eta}\frac{\eta\,\partial U_{z0}}{\partial \eta}\right]+4\,\frac{\partial^2 U_{z0}}{\partial \zeta^2}=0. \tag{2.12}$$

This is to be solved with the boundary conditions and initial conditions

$$U_{z0}=0 \quad \text{at } \zeta=1, \tag{2.13}$$

$$U_{z0}=W_{\mathrm{B}}(\eta)\,F(T/T_{\mathrm{s}}) \quad \text{at } \zeta=0, \quad \text{all } T, \tag{2.14}$$

$$U_{z0}=0 \quad \text{at } T=0. \tag{2.15}$$

The other velocity components are then determined from (2.5) and (2.6), whence

$$U_{r0}=-\frac{1}{\eta}\int_0^{\eta}\frac{\partial U_{z0}}{\partial \zeta}\,\eta\,\mathrm{d}\eta, \tag{2.16}$$

$$U_{\theta 1}=-\frac{1}{2}\int_0^{T} U_{r0}\,\mathrm{d}T. \tag{2.17}$$

The problem posed above may be solved by straightforward application of linear analysis which is outlined in Appendix A. The solution is

$$U_{z0}=W_{\mathrm{B}}(\eta)\,(1-\zeta)\,F(T/T_{\mathrm{s}})+\sum_{n=1}^{\infty}\int_0^{\infty}\tilde{U}_n(\lambda,T)\,J_0(\lambda\eta)\,\lambda\,\mathrm{d}\lambda\,\sin n\pi(1-\zeta), \tag{2.18}$$

where $\tilde{U}_n$ is given by (A 11) in the appendix.

The first term in (2.18) is the same as the instantaneous suction profile scaled by $(1-\zeta)$ while the second consists of inertial oscillations (a superposition of spatial modes times trigonometric functions of time). In the absence of viscosity these low-frequency oscillations do not decay out in time and in general are not small. For instance, if the suction is suddenly imposed and then held steady the inertial oscillations are of the same order as the suction velocity. However, if the suction is slowly imposed these oscillations may be made arbitrarily small because the time derivative in (A 11) may be written

$$\frac{\mathrm{d}^2F}{\mathrm{d}T^2}=\frac{F''}{T_{\mathrm{s}}^2}, \tag{2.19}$$

and is small if T_{s} is large enough. The primes indicate derivatives with respect to the argument T/T_{s}.

The other velocity components are determined from (2.16) and (2.17) and the results are thus

$$U_{z0} = W_B(\eta)\,(1-\zeta)\,F(T/T_s) + \text{inertial oscil.}, \tag{2.20}$$

$$U_{r0} = \frac{1}{\eta}\int_0^\eta W_B(\eta)\,\eta\,\mathrm{d}\eta\,F(T/T_s) + \text{inertial oscil.}, \tag{2.21}$$

$$U_{\theta 1} = -\frac{1}{2}\,\frac{1}{\eta}\int_0^\eta W_B(\eta)\,\eta\,\mathrm{d}\eta\int_0^T F(T/T_s)\,\mathrm{d}T + \text{inertial oscil.} \tag{2.22}$$

Two things should be noticed. Except for the inertial oscillations U_{z0} and U_{r0} are both quasi-steady, being scaled with the instantaneous mean suction velocity and while $U_{\theta 1}$ has the same radial dependence as U_{r0} it continually grows with time. This means that eventually the perturbation swirling velocity becomes comparable to the initial solid-body rotation and the expansion becomes invalid.

3. The nonlinear inviscid problem

Motivated by the form of the linear solution the radial and vertical velocity components will be rescaled by the *instantaneous* average velocity at the hole by defining

$$W = \frac{U_z}{F(T/T_s)} = \frac{u_z}{u_\infty F(T/T_s)}, \tag{3.1}$$

$$V = \frac{U_r}{F(T/T_s)} = \frac{u_r}{u_\infty F(T/T_s)}. \tag{3.2}$$

A slow time will be defined by

$$\tau = \epsilon\int_0^T F(T/T_s)\,\mathrm{d}T; \tag{3.3}$$

which tends to $\tau = \epsilon T$ if T is large. This is the total volume flow through the drain up to time t, divided by $\pi R^2 H$, the volume above the hole. It will be assumed that the inertial oscillations, which have a different timescale, occur only in higher-order terms of a perturbation expansion. The lowest-order terms must therefore satisfy

$$\frac{1}{\eta}\frac{\partial \eta V}{\partial \eta} + \frac{\partial W}{\partial \zeta} = 0, \tag{3.4}$$

$$\left[\frac{\partial}{\partial \tau} + V\frac{\partial}{\partial \eta} + W\frac{\partial}{\partial \zeta}\right]\eta U_\theta = 0, \tag{3.5}$$

$$-\frac{U_\theta^2}{\eta} = -\frac{\partial P}{\partial \eta}, \tag{3.6}$$

$$0 = -\frac{\partial P}{\partial \zeta} - G, \tag{3.7}$$

with boundary and initial conditions

$$W = 0 \quad \text{at } \zeta = 1, \tag{3.8}$$

$$W = W_B(\eta) \quad \text{at } \zeta = 0, \tag{3.9}$$

$$U_\theta = \eta \quad \text{at } \tau = 0. \tag{3.10}$$

The problem described in the above equations may be considerably simplified. From (3.7) it is clear the $\partial P/\partial \eta$ is independent of ζ, therefore (3.6) implies that U_θ is also independent of ζ and (3.5) may thus be written

$$\left[\frac{\partial}{\partial \tau}+V\frac{\partial}{\partial \eta}\right]\eta U_\theta = 0. \tag{3.11}$$

In order for this to be consistent V must also be independent of ζ. Since V is independent of ζ, (3.4) implies that W can be no more than linear in ζ. In light of the boundary condition W is therefore given by

$$W = W_{\mathrm{B}}(\eta)\,(1-\zeta), \tag{3.12}$$

and therefore

$$V = \frac{1}{\eta}\int_0^\eta W_{\mathrm{B}}(\eta)\,\eta\,\mathrm{d}\eta. \tag{3.13}$$

The problem thus reduces to solving (3.11), with V given by (3.13), subject to the initial condition $U_\theta = \eta$.

Since W_{B} is defined to be zero for $\eta > 1$ and has unit average over the hole, (3.13) gives

$$V = -\frac{1}{2\eta}, \tag{3.14}$$

for $\eta > 1$ and depends on the details of the profile for $\eta < 1$. For uniform flow $(W_{\mathrm{B}} = -1)$

$$V = -\tfrac{1}{2}\eta, \quad \eta < 1, \tag{3.15}$$

while, for a parabolic profile $[W_{\mathrm{B}} = -2(1-\eta^2)]$,

$$V = -\eta(1-\tfrac{1}{2}\eta^2), \quad \eta < 1. \tag{3.16}$$

Equation (3.12) may be solved by the method of characteristics, which is equivalent here to integrating along particle paths. The equation may be written as a pair of ordinary differential equations,

$$\frac{\mathrm{d}\eta U_\theta}{\mathrm{d}\tau} = 0, \tag{3.17}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = V(\eta), \tag{3.18}$$

with initial conditions

$$\eta = \xi, \quad U_\theta = \xi \quad \text{at } \tau = 0, \tag{3.19}$$

where ξ, the initial position of a particle, allows a parametric representation of the solution. The parametric solution is

$$\eta U_\theta = \xi^2, \tag{3.20}$$

$$\int_1^\eta \frac{\mathrm{d}\eta}{V(\eta)} = \tau + \int_1^\xi \frac{\mathrm{d}\eta}{V(\eta)}. \tag{3.21}$$

The integrations in (3.21) are to be carried out, solved for ξ as a function of η and τ which is then to be substituted into (3.20).

These operations are easily carried out for uniform suction flow with V given by (3.14) and (3.15). The results are

$$\left.\begin{aligned} \eta U_\theta &= \eta^2 \mathrm{e}^\tau, \quad \eta < \mathrm{e}^{-\frac{1}{2}\tau}; \\ &= 1+\tau+\ln(\eta^2), \quad \mathrm{e}^{-\frac{1}{2}\tau} < \eta < 1; \\ &= \tau+\eta^2, \quad \eta > 1. \end{aligned}\right\} \tag{3.22}$$

The part for $\eta > 1$ is universal for any suction profile and consists of a solid-body rotation plus a potential vortex whose strength increases linearly with time. The rest of the profile depends on the particular suction profile.

The vorticity distribution is also interesting. This may be calculated from

$$\frac{\omega_z}{\Omega} = \frac{1}{\eta}\frac{\partial \eta U_\theta}{\partial \eta}, \tag{3.23}$$

and is found to be

$$\left.\begin{aligned} \frac{\omega_z}{\Omega} &= 2\mathrm{e}^\tau, \quad \eta < \mathrm{e}^{-\frac{1}{2}\tau}; \\ &= \frac{2}{\eta^2}, \quad \mathrm{e}^{-\frac{1}{2}\tau} < \eta < 1; \\ &= 2, \quad \eta > 1. \end{aligned}\right\} \tag{3.24}$$

This shows that the amplification of the peak vorticity over the background vorticity goes like e^τ. Notice that the vorticity of the innermost region changes with time and therefore the solution never reaches a steady state. This happens because as time increases the fluid above the hole has come from ever greater distances and possesses the circulation appropriate for that location.

Similar results are found for the parabolic suction profile, with V given by (3.14) and (3.16). The results are

$$\left.\begin{aligned} \eta U_\theta &= \frac{2\eta^2}{2\exp(-2\tau)+[1-\exp(-2\tau)]\eta^2}, \quad \eta^2 < \frac{2}{1+\exp(2\tau)}; \\ &= 1+\tau-\tfrac{1}{2}\ln\frac{2-\eta^2}{\eta^2}, \quad \frac{2}{1+\exp(2\tau)} < \eta^2 < 1; \\ &= \tau+\eta^2, \quad \eta > 1; \end{aligned}\right\} \tag{3.25}$$

and

$$\left.\begin{aligned} \frac{\omega_z}{\Omega} &= \frac{8\exp(2\tau)}{[2+[\exp(2\tau)-1]\eta^2]^2}, \quad \eta^2 < \frac{2}{1+\exp(2\tau)}; \\ &= \frac{2}{\eta^2(2-\eta^2)}, \quad \frac{2}{1+\exp(2\tau)} < \eta^2 < 1; \\ &= 2, \quad \eta > 1. \end{aligned}\right\} \tag{3.26}$$

In this case there is a much more intense concentration of vorticity near the axis.

A variation on the problem considered above is the transient version of the Einstein & Li problem discussed in the introduction. The fluid is initially rotating as a rigid body when the drain is opened as before but fluid which enters through a cylindrical boundary at $\eta = \eta_1 = R_1/R$ has steady angular velocity of the same value as the initial rotation. Mathematically the problem is to solve

$$\left[\frac{\partial}{\partial \tau} + V\frac{\partial}{\partial \eta}\right]\eta U_\theta = 0, \tag{3.27}$$

with
$$U_\theta = \eta \quad \text{at } \tau = 0, \quad \eta < \eta_1, \tag{3.28}$$
$$U_\theta = \eta_1 \quad \text{at } \eta = \eta_1, \quad \text{all } \tau. \tag{3.29}$$

The solution is easily found. All the fluid which is initially within the container spins up in exactly the same manner as for the infinite case, while all the fluid which enters through the side boundary has
$$\eta U_\theta = \eta_1^2. \tag{3.30}$$

The particle path which separates the two solutions satisfies
$$\frac{d\eta}{d\tau} = V(\eta), \quad \eta = \eta_1 \quad \text{at } \tau = 0. \tag{3.31}$$

For the uniform suction profile the separating path is
$$\left.\begin{aligned} \eta &= (\eta_1^2 - \tau)^{\frac{1}{2}}, \quad \tau < \eta_1^2 - 1; \\ &= e^{\frac{1}{2}(\eta_1^2 - 1 - \tau)}, \quad \tau > \eta_1^2 - 1, \end{aligned}\right\} \tag{3.32}$$

and therefore the solution is:

(i) $\tau < \eta_1^2 - 1$,
$$\left.\begin{aligned} \eta U_\theta &= \eta^2 e^\tau, \quad \eta < e^{-\frac{1}{2}\tau}; \\ &= 1 + \tau + \ln(\eta^2), \quad e^{-\frac{1}{2}\tau} < \eta < 1; \\ &= \tau + \eta^2, \quad 1 < \eta < (\eta_1^2 - \tau)^{\frac{1}{2}}; \\ &= \eta_1^2, \quad (\eta_1^2 - \tau)^{\frac{1}{2}} < \eta < \eta_1; \end{aligned}\right\} \tag{3.33}$$

(ii) $\tau > \eta_1^2 - 1$,
$$\left.\begin{aligned} \eta U_\theta &= \eta^2 e^\tau, \quad \eta < e^{-\frac{1}{2}\tau}; \\ &= 1 + \tau + \ln(\eta^2), \quad e^{-\frac{1}{2}\tau} < \eta < e^{\frac{1}{2}(\eta_1^2 - 1 - \tau)}; \\ &= \eta_1^2, \quad e^{\frac{1}{2}(\eta_1^2 - 1 - \tau)} < \eta < \eta_1. \end{aligned}\right\} \tag{3.34}$$

In general, for any suction profile, the infinite-domain solution is valid as long as ηU_θ remains smaller than η_1^2: wherever ηU_θ begins to poke above η_1^2 it must be replaced by η_1^2. Similarly the vorticity must be replaced by zero wherever ηU_θ is replaced by η_1^2.

4. The effect of interior viscosity

In the problem considered in the previous sections the swirling velocity continues to increase as vorticity becomes more intense along the axis. At first sight one might expect internal viscosity to ultimately counter this effect: however this is not the case. The appropriate equation is (1.7) with the viscous terms retained. This may be written
$$\left[\frac{\partial}{\partial \tau} + V \frac{\partial}{\partial \eta}\right] \eta U_\theta = N \left[\frac{\partial^2 \eta U_\theta}{\partial \eta^2} - \frac{1}{\eta} \frac{\partial \eta U_\theta}{\partial \eta}\right], \tag{4.1}$$

where the parameter
$$N = \frac{\nu H}{u_\infty R^2} \tag{4.2}$$

will be assumed to be small and V is the same function used previously. A solution is sought for large time which matches the inviscid solution for small time. It is convenient to convert this equation into a vorticity equation by using
$$\frac{\omega_z}{\Omega} = \frac{1}{\eta} \frac{\partial \eta U_\theta}{\partial \eta}, \tag{4.3}$$

and operating on (4.1) with $\eta^{-1}\,\partial/\partial\eta$. This gives

$$\left[\frac{\partial}{\partial\tau}+V\frac{\partial}{\partial\eta}\right]\frac{\omega_z}{\Omega}=-W_{\mathrm{B}}(\eta)\frac{\omega_z}{\Omega}+N\left[\frac{\partial^2}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\right]\frac{\omega_z}{\Omega}. \tag{4.4}$$

The first term on the right is the vorticity-production term (i.e. $\boldsymbol{\omega}\cdot\nabla u$).

Further discussion will be limited to the uniform-suction case where $W_{\mathrm{B}}=-1$ when $\eta<1$ and $W_{\mathrm{B}}=0$ when $\eta>1$ and the inviscid solution is given by (3.24). When the inviscid solution is substituted into the neglected viscous terms it becomes clear that these terms are important when $\eta\approx N^{\frac{1}{2}}$ and the inviscid solution remains valid for values much larger than this. Since $\exp(-\frac{1}{2}\tau)$ is a measure of the width of the vortex core the viscous terms become important when $N^{\frac{1}{2}}\approx\exp(-\frac{1}{2}\tau)$, that is, at modestly large times of order $\ln(N^{-1})$.

Motivated by the above discussion let a new radial variable be defined by

$$s=\eta/N^{\frac{1}{2}}, \tag{4.5}$$

and restrict attention to the part of the flow where $\eta<1$ (since $\eta=1$ occurs at $s=N^{-\frac{1}{2}}$, which tends to infinity as $N\to 0$). The equation becomes

$$\left[\frac{\partial}{\partial\tau}-\tfrac{1}{2}s\frac{\partial}{\partial s}\right]\frac{\omega_z}{\Omega}=\frac{\omega_z}{\Omega}+\left[\frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}\right]\frac{\omega_z}{\Omega}. \tag{4.6}$$

The solution should match the inviscid solution for large s. Comparing with (3.25) this condition is seen to be

$$N\frac{\omega_z}{\Omega}\to\frac{2}{s^2}\quad\text{as } s\to\infty. \tag{4.7}$$

The asymptotic solution to this problem for large time is developed in Appendix B. It is found there that

$$N\frac{\omega_z}{\Omega}=(C_2+\tfrac{1}{2}\tau)\,\mathrm{e}^{-\frac{1}{4}s^2}+\mathrm{e}^{-\frac{1}{4}s^2}\int_0^s\frac{\mathrm{e}^{\frac{1}{4}s^2}-1}{s}\,\mathrm{d}s, \tag{4.8}$$

with $$C_2=\tfrac{1}{2}\ln(N)+0.9045.$$

When τ is large the first term dominates. This is like Burgers' vortex, but with a circulation which increases with time. Thus, while interior viscosity modifies the velocity profile, the solution continues to swirl faster as time increases but not as fast as in the inviscid case. The maximum vorticity grows linearly with time compared with the exponential growth found for the inviscid flow.

It remains to discuss the Einstein & Li version of this problem. In this case the circulation about a distant contour remains constant and ultimately the viscous problem will have a steady solution. The inviscid vorticity distribution from the last section, for uniform suction, is

$$\left.\begin{aligned}\frac{\omega_z}{\Omega}&=2\mathrm{e}^{\tau},\quad \eta<\mathrm{e}^{-\frac{1}{2}\tau};\\ &=\frac{2}{\eta^2},\quad \mathrm{e}^{-\frac{1}{2}\tau}<\eta<\mathrm{e}^{\frac{1}{2}(\eta_1^2-1-\tau)};\\ &=0,\quad \mathrm{e}^{\frac{1}{2}(\eta_1^2-1-\tau)}<\eta<\eta_1.\end{aligned}\right\} \tag{4.9}$$

This is for $\tau>\eta_1^2-1$. For a long time the effect of viscosity is the same as the infinite-domain case just analysed, with the viscous solution matching into the $2/\eta^2$ part of the inviscid vorticity. The width of this viscous region is of order $N^{\frac{1}{2}}$

and stays constant while a second viscous layer which spreads the discontinuity at $\eta = \exp[\frac{1}{2}(\eta_1^2 - 1 - \tau)]$ moves radially inward. These two layers merge when $\tau - (\eta_1^2 - 1) = O(\ln N^{-1})$, which could be a large time if η_1 is fairly large (for example, 10). The merged layers finally evolve into a steady vorticity distribution because the total vorticity must stay the same as in the inviscid solution. That is, A from (9.4) stays constant because the vorticity outside the second layer drops rapidly to zero. The steady solution must satisfy (4.6) with the integral condition

$$\int_0^\infty N \frac{\omega_z}{\Omega} s \, \mathrm{d}s = \eta_1^2, \tag{4.10}$$

and is easily found to be

$$\frac{\omega_z}{\Omega} = \frac{\eta_1^2}{2N} \mathrm{e}^{-\frac{1}{4}s^2} = \frac{\eta_1^2}{2N} \mathrm{e}^{-\eta^2/4N}, \tag{4.11}$$

$$\eta U_\theta = \eta_1^2 (1 - \mathrm{e}^{-\eta^2/4N}). \tag{4.12}$$

This is exactly the same as Burgers' (1948) vortex solution. (The Einstein–Li solution is more general than this because they do not restrict it to small N; consequently there is another form of the solution in the region $\eta > 1$. The differences tend to zero when $N \to 0$.) Burgers' vortex is the steady solution of a viscous vortex with finite circulation in an axially straining velocity field. This is exactly the physical situation in the region above the drain hole because of the linear variation of the axial velocity.

It is interesting to consider the effect of free-surface variation on this last problem. The flow will be assumed steady with the Einstein–Li boundary condition, at $\eta = \eta_1$ and uniform suction at the drain hole. The elevation of the free surface will be given by $\zeta = h(\eta)$ with the depth specified to be unity at the position $\eta = \eta_1$, i.e. $h(\eta_1) = 1$. This means that the vertical lengthscale H is the depth at this location. The same reasoning as before shows that U_θ and V are independent of ζ and that W varies linearly with ζ; however W is not zero at the upper surface but is related to V by

$$\frac{W}{V} = \frac{\mathrm{d}h}{\mathrm{d}\eta} \quad \text{at } \zeta = h(\eta), \tag{4.13}$$

since the free surface is a streamline. The continuity equation may be written

$$\frac{1}{\eta} \frac{\partial}{\partial \eta} \eta V + \frac{W(\zeta = h) - W(\zeta = 0)}{h} = 0, \tag{4.14}$$

and using (4.13) to eliminate $W(\zeta = h)$ gives

$$\frac{1}{\eta} \frac{\partial}{\partial \eta} h \eta V = W(\zeta = 0). \tag{4.15}$$

For uniform suction this may be integrated to give

$$\left.\begin{aligned} hV &= -\tfrac{1}{2}\eta, \quad \eta < 1; \\ &= -\frac{1}{2}\frac{1}{\eta}, \quad \eta > 1; \end{aligned}\right\} \tag{4.16}$$

which has a clear physical meaning.

At the free surface the pressure is constant, therefore (3.7) gives

$$P = G(h - \zeta), \tag{4.17}$$

and (3.6) relates h and U_θ;

$$G \frac{\mathrm{d}h}{\mathrm{d}\eta} = \frac{U_\theta^2}{\eta}. \tag{4.18}$$

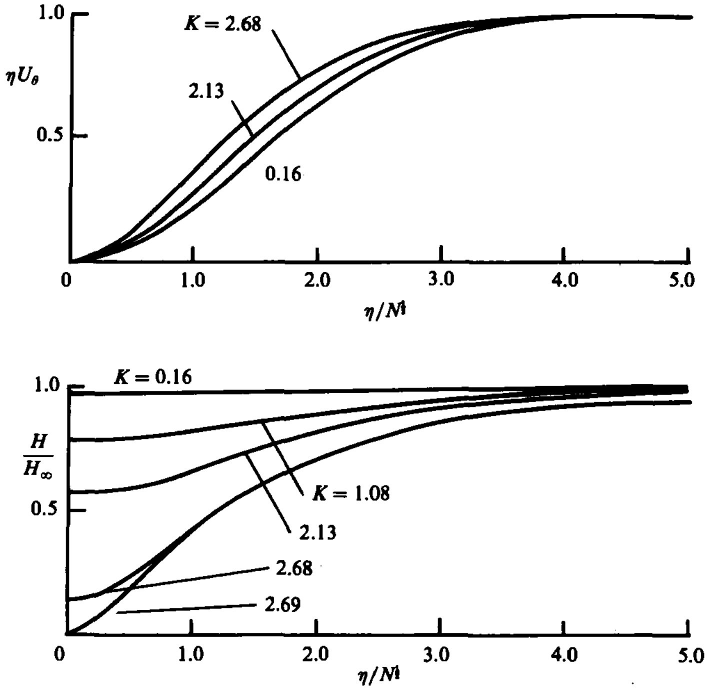


FIGURE 1. Circulation distribution and height of free surface in a rotating cylinder with uniform outflow at the drain. K and N are defined in §4.

The problem is thus to solve (4.18) and (4.1) with V given by (4.16). When N is zero these equations may be solved for U_θ and h;

$$\eta U_\theta = \eta_1^2, \tag{4.19}$$

$$h = 1 - \frac{\eta_1^4}{2G}\left[\frac{1}{\eta^2} - \frac{1}{\eta_1^2}\right]. \tag{4.20}$$

This is the classical solution of the free surface induced by a potential vortex and represents the case where the air core plunges through the drain hole. Of course this violates the condition of uniform suction velocity but is a valid solution for a more general problem since it doesn't depend on assumptions about the suction velocity. However, when N is not zero, (4.1) depends explicitly on V and therefore on the prescribed suction velocity and the solution must therefore be restricted to positive h (or else the boundary conditions must be changed).

When N is small the viscous terms become important when η is of order $N^{\frac{1}{2}}$. Therefore (4.10) suggests that the parameters should be restricted to a range where

$$K = \frac{\eta_1^4}{GN}, \tag{4.21}$$

is of order one. A change of variables to

$$g = \frac{\eta U_\theta}{\eta_1^2}, \tag{4.22}$$

$$s = \eta/N^{\frac{1}{2}}, \tag{4.23}$$

gives the pair of equations

$$-\frac{1}{2}\frac{s}{h}\frac{\mathrm{d}g}{\mathrm{d}s}=\frac{\mathrm{d}^2g}{\mathrm{d}s^2}-\frac{1}{s}\frac{\mathrm{d}g}{\mathrm{d}s}, \tag{4.24}$$

$$\frac{\mathrm{d}h}{\mathrm{d}s}=K\frac{g^2}{s^3}, \tag{4.25}$$

which must be solved with the asymptotic boundary conditions $g \rightarrow 1$, $h \rightarrow 1$ as $s \rightarrow \infty$ and $g = 0$ at $s = 0$. These equations have been solved numerically with the results shown in figure 1. When K is small the velocity distribution tends to Burgers' solution and doesn't deviate much from this for any allowable value of K. When K tends to the value 2.69 the depth at the centre tends rapidly to zero and there is no allowable solution for larger K. This would seem to mean that when the drain plumbing is specified in some realistic way there is a critical value of K probably near 2.69 at which the air core passes through the hole. Rott (1958) discusses the rapid breakthrough of the air core with increasing circulation, identifying a parameter similar to K.

5. The effect of viscosity at the boundaries

It is assumed that the effect is confined to thin boundary layers and vertical shear layers. For linearized problems the theory of the Ekman boundary layers is well developed. In the inviscid interior a balance between Coriolis forces and pressures forces provides circumferential velocity in response to a radial pressure gradient. However, near a boundary the Coriolis force is decreased by viscosity and the same pressure forces induce strong radial flow in a boundary layer with thickness of order $(\nu/\Omega)^{\frac{1}{2}}$. Conservation of mass in this layer produces a vertical velocity at the outer edge of the Ekman layer in a form suitable for use as a boundary condition for the interior flow. An approximate boundary condition for nonlinear Ekman layers has been used by Turner (1966) for tornado-like flows similar to those of this paper. This makes use of a numerical analysis of the flow between rotating disks by Rogers & Lance (1960). A similar use was made by Wedemeyer (1964) in a nonlinear spin-up problem. The result is that the radial flux per unit length in the Ekman layer is approximated by

$$M=-0.67(u_\theta-r\Omega)\,r\left(\frac{\nu}{ru_\theta}\right)^{\frac{1}{2}},$$

where $r\Omega$ is the velocity of the boundary and u_θ the velocity in the interior ($2\pi rM$ is the total radial flux in the layer). Conservation of mass gives the velocity at the edge of the layer,

$$U_z=\pm\frac{1}{r}\frac{\partial rM}{\partial r}. \tag{5.1}$$

with the plus sign at an upper boundary, the minus at a lower boundary. When put in dimensionless form this becomes

$$W=\pm\phi\frac{1}{\eta}\frac{\partial}{\partial\eta}\frac{\eta(\eta U_\theta-\eta^2)}{(\eta U_\eta)^{\frac{1}{2}}}, \tag{5.2}$$

where
$$\phi=0.67\frac{(\nu\Omega)^{\frac{1}{2}}}{u_\infty}. \tag{5.3}$$

This parameter may also be written

$$\phi = 0.67\,\frac{E^{\frac{1}{2}}}{\epsilon}, \tag{5.4}$$

where $E = \nu/\Omega H^2$ is an Ekman number. The parameter ϕ is not necessarily small.

5.1. *The effect of viscosity at the lower boundary*

The problem of slow efflux will be considered again with friction allowed on the lower boundary, the upper boundary being frictionless and flat; an idealized free surface. The problem is the same as in §3 except that the velocity at the lower surface is given by (5.2) when $\eta > 1$. For uniform suction the boundary condition is given by

$$W(\zeta = 0) = -1, \quad \eta < 1; \tag{5.6}$$

$$= \phi\,\frac{1}{\eta}\,\frac{\partial}{\partial\eta}\,\frac{\eta(\eta U_\theta - \eta^2)}{(\eta U_\theta)^{\frac{1}{2}}}, \quad \eta > 1. \tag{5.7}$$

Since W is a linear function of ζ, as before, one finds

$$\frac{1}{\eta}\,\frac{\partial}{\partial\eta}\,\eta V = W(\zeta = 0), \tag{5.8}$$

from which

$$V = -\tfrac{1}{2}\eta, \quad \eta < 1; \tag{5.9}$$

$$= \phi\,\frac{(\eta U_\theta - \eta^2)}{(\eta U_\theta)^{\frac{1}{2}}} - \frac{1}{2}\,\frac{1}{\eta}, \quad \eta > 1. \tag{5.10}$$

The first term on the right is the negative of the radial flux per unit length carried in the Ekman layer. At the initial time this term is zero. Since ηU_θ becomes larger than η^2 as time proceeds, more of the radial inflow is carried by the Ekman layer and less by the interior velocity. Except for a special case a discontinuity in V will develop at $\eta = 1$ with a greater inflow required for $\eta < 1$ than can be supplied by the interior flow from $\eta > 1$. The required flow is supplied from a vertical shear layer at $\eta = 1$ which carries a thin stream of fluid upward from the Ekman layer. The linear theory of vertical shear layers is well developed (see Greenspan 1968) and it is known that such layers have a complex double-layer structure. A layer with thickness of order $E^{\frac{1}{4}}$ smooths out corners in the U_θ profile while a thinner embedded layer with thickness of order $E^{\frac{1}{3}}$ carries a vertical flux of order $E^{\frac{1}{2}}$ (it is of order ϕ $(= E^{\frac{1}{2}}/\epsilon)$ in the present dimensionless scheme). The structural details of these layers are not needed in this paper but are assumed to be similar to those described above. Simple experiments by Lewellen (1962) verify the existence of thin vertical shear layers near the drain hole with flow in a direction opposite to that through the hole.

The mathematical problem of solving

$$\frac{\partial\eta U_\theta}{\partial\tau} + V\,\frac{\partial\eta U_\theta}{\partial\eta} = 0, \tag{5.11}$$

with V given by (5.9) and (5.10) and with $\eta U_\theta = \eta^2$ at $\tau = 0$, is equivalent to solving

$$\left.\begin{aligned} \frac{d\eta}{d\tau} = V, \quad \eta = \xi \quad \text{at } \tau = 0; \\ \eta U_\theta = \xi^2, \end{aligned}\right\} \tag{5.12}$$

as before, except that the expression for V also depends on ξ. The parametric solution is formally the same;

$$\left.\begin{aligned}\int_1^\eta \frac{d\eta}{V(\eta,\xi)} &= \tau + \int_1^\xi \frac{d\eta}{V(\eta,\xi)},\\ \eta U_\theta &= \xi^2.\end{aligned}\right\} \tag{5.13}$$

Depending on whether η and ξ are less than, or greater than, one this breaks down into three expressions;

(i) $$\eta U_\theta = \xi^2 = \eta^2 e^\tau, \quad \eta < e^{-\frac{1}{2}\tau}; \tag{5.14}$$

(ii) $$\left.\begin{aligned} -\ln(\eta^2) &= \tau + \int_1^\xi \frac{\alpha\, d\alpha}{\phi(\xi^2-\alpha^2)\alpha/\xi - \frac{1}{2}}, \quad e^{-\frac{1}{2}\tau} < \eta < 1, \quad \xi > 1;\\ \eta U_\theta &= \xi^2;\end{aligned}\right\} \tag{5.15}$$

(iii) $$\left.\begin{aligned} 0 &= \tau + \int_\eta^\xi \frac{\alpha\, d\alpha}{\phi(\xi^2-\alpha^2)\alpha/\xi - \frac{1}{2}}, \quad \eta > 1, \quad \xi > \eta;\\ \eta U_\theta &= \xi^2.\end{aligned}\right\} \tag{5.16}$$

In (ii) and (iii) the integrations cannot be carried out easily in closed form. The solution will be analysed by small- and large-time expansions supplemented by numerical integration.

Region (iii), $\eta > 1$, will be considered first. When $\tau \to 0$, $\xi \to \eta$; therefore a change of variables $\xi = \eta + \Delta$, $\alpha = \eta + \Delta h$ will be made, with h a new integration variable and Δ small. A straightforward expansion in powers of Δ gives

$$\tau = 2\eta\Delta + (1 - 4\phi\eta^2)\Delta^2 + \dots. \tag{5.17}$$

Solving this for Δ one finds

$$\Delta = \frac{\tau}{2\eta} - (1-4\phi\eta^2)\frac{\tau^2}{8\eta^3} + O(\tau^3), \tag{5.18}$$

and hence $$\eta U_\theta = \xi^2 = (\eta+\Delta)^2 = \eta^2 + \tau + \phi\tau^2 + O(\tau^3). \tag{5.19}$$

Comparing this with (3.23) it is seen that bottom friction causes the angular velocity to become intensified. The radial velocity is calculated from (5.10) giving

$$V = -\frac{1}{2}\frac{1}{\eta} + \frac{\phi}{\eta}\tau - \frac{\phi}{\eta}\left[\phi - \frac{1}{2\eta^2}\right]\tau^2 + O(\tau^3). \tag{5.20}$$

The lowest-order correction, $\phi\tau/\eta$, decreases the radial inflow in the interior which must be compensated by a radial inflow carried by the Ekman layer, which thus consists of a finite total Ekman flux from infinity. This will be shown later to make sense in terms of an Einstein–Li boundary condition at large radius. The vertical velocity at the edge of the Ekman layer is calculated from (5.8) and is found to be

$$W(\zeta = 0) = \frac{\phi\tau^2}{\eta^4} + O(\tau^3), \tag{5.21}$$

since the first two terms in (5.20) don't contribute. This is seen to be positive, increasing for smaller radius. The picture is thus presented as a radial inflow from infinity, part of which is carried in an Ekman layer from which a portion flows upwards into the interior.

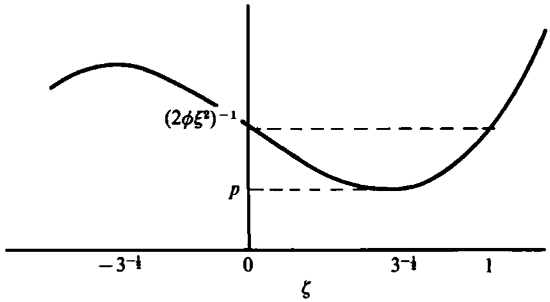


FIGURE 2. Sketch of the cubic polynomial which occurs in the denominator of (5.22).

A similar conclusion is reached by expanding for large time. Consider (5.16) again. With a change of variable to $\alpha = \xi\zeta$ this may be written

$$\phi\tau = \int_{\eta/\xi}^{1} \frac{\zeta\, d\zeta}{\zeta^3 - \zeta + (2\phi\xi^2)^{-1}}. \tag{5.22}$$

The cubic in the denominator is sketched in figure 2. It has a minimum at $\zeta = 3^{-\frac{1}{2}}$ with height $p = (2\phi\xi^2)^{-1} - (2)\,3^{-\frac{3}{2}}$. As $\tau \to \infty$ the integral must become singular. When $\eta/\xi < 3^{-\frac{1}{2}}$ this can only occur if the minimum tends to the axis, i.e. $p \to 0$. When $\eta/\xi > 3^{-\frac{1}{2}}$ it occurs with the minimum at a negative value and with the lower limit of integration, η/ξ, tending to the right-hand zero crossing. That is

$$\left[\frac{\eta}{\xi}\right]^3 - \frac{\eta}{\xi} + \frac{1}{2\phi\xi^2} = 0, \tag{5.23}$$

or

$$\xi^2 - \frac{\xi}{2\phi\eta} - \eta^2 = 0. \tag{5.24}$$

Therefore it is clear that as $\tau \to \infty$ a steady-state solution is reached for $\eta > 1$. Solving for ξ in the above expressions gives this solution as

$$\eta U_\theta = \xi^2 = \frac{3^{\frac{3}{2}}}{4\phi}, \quad \eta < \frac{3^{\frac{1}{4}}}{2\phi^{\frac{1}{2}}}; \tag{5.25}$$

$$\eta U_\theta = \xi^2 = [(4\phi\eta)^{-1} + (\eta^2 + (4\phi\eta)^{-2})^{\frac{1}{2}}]^2, \quad \eta > \frac{3^{\frac{1}{4}}}{2\phi^{\frac{1}{2}}}. \tag{5.26}$$

If $\phi > 3^{\frac{1}{2}}/4$ only (5.26) occurs. The inner part of this solution is a potential vortex while the outer part makes a transition to solid-body rotation at infinity. This result is plotted in figure 5 for a particular value of ϕ. The radial velocity, from (5.20), is

$$V = -\frac{1}{2}\,\frac{1}{\eta} + \frac{\phi(3^{\frac{3}{2}}/4\phi - \eta^2)}{3^{\frac{3}{4}}/2\phi^{\frac{1}{2}}}, \quad \eta < \frac{3^{\frac{1}{4}}}{2\phi^{\frac{1}{2}}}; \tag{5.27}$$

$$V = 0, \quad \eta > \frac{3^{\frac{1}{4}}}{2\phi^{\frac{1}{2}}}. \tag{5.28}$$

The last form occurs because (5.20) and (5.24) are the same. The radial velocity is zero throughout the region $\eta > 1$ if $\phi > 3^{\frac{1}{2}}/4$. When V is zero the total radial flow is carried in the Ekman layer.

Finally, the vertical velocity is

$$W(\zeta = 0) = 2 \times 3^{\frac{1}{4}} \phi^{\frac{3}{2}} \left(\frac{3^{\frac{1}{2}}}{4\phi} - \eta^2 \right) \Big/ \eta, \quad \eta < \frac{3^{\frac{1}{4}}}{2\phi^{\frac{1}{2}}}; \tag{5.29}$$

$$= 0, \quad \eta > \frac{3^{\frac{1}{4}}}{2\phi^{\frac{1}{2}}}. \tag{5.30}$$

For $\eta < 3^{\frac{1}{4}}/2\phi^{\frac{1}{2}}$ there is some upward flow from the Ekman layer into the interior. However if $\phi > 3^{\frac{1}{2}}/4$ the inner region does not occur and $W(\zeta = 0)$ is zero throughout. The total flow from the Ekman layer must then flow upward in the vertical shear layer at $\eta = 1$.

Notice that for a steady state to exist, (5.11) says that either $V = 0$ or ηU_θ is constant. This is exactly the solution which has been found. Without going through the transient analysis it would not be clear that the part of the solution which has constant ηU_θ extends from the minimum of the ηU_θ function calculated from $V = 0$.

The problem of flow from infinity in the Ekman layer can be better understood by considering the Einstein–Li boundary condition at a finite radius;

$$U_\theta = \eta_1, \quad V = -\frac{1}{2}\frac{1}{\eta_1} \tag{5.31}$$

at $\eta = \eta_1 > 1$. The fluid is forced to enter uniformly through the cylindrical sidewall. The part of the fluid which was in the container at $\tau = 0$ evolves as before. However, the fluid which enters through the sidewall reaches a steady state right away with

$$\left.\begin{aligned} \eta U_\theta &= \eta_1^2, \\ V &= \frac{\phi(\eta_1^2 - \eta^2)}{\eta_1} - \frac{1}{2}\frac{1}{\eta}, \\ W(\zeta = 0) &= \phi\left[\frac{\eta_1}{\eta} - 3\frac{\eta}{\eta_1}\right] \quad \text{for } \eta_{\mathrm{F}}(\tau) < \eta < \eta_1, \end{aligned}\right\} \tag{5.32}$$

where $\eta_{\mathrm{F}}(\tau)$, the location of the front of the incoming fluid, must satisfy

$$\frac{d\eta_{\mathrm{F}}(\tau)}{d\tau} = V(\eta_{\mathrm{F}}), \quad \eta_{\mathrm{F}} = \eta_1 \quad \text{at } \tau = 0. \tag{5.33}$$

There are two situations which can occur. If V is negative throughout the region $1 < \eta < \eta_1$, the front will reach $\eta = 1$ in the finite time

$$\tau_{\mathrm{F}} = -\int_1^{\eta_1} \frac{d\eta}{V}. \tag{5.34}$$

Since the cubic expression for $-\eta V$ has a minimum at $\eta = \eta_1/\sqrt{3}$, negative V will occur when $\eta_1 < 3^{\frac{3}{4}}/2\phi^{\frac{1}{2}}$ if $\phi < \frac{1}{4}3^{\frac{1}{2}}$ (which ensures that the minimum occurs for $\eta > 1$), or when $\eta_1 < (4\phi)^{-1} + (1 + (4\phi)^{-2})^{\frac{1}{2}}$ if $\phi > \frac{1}{4}3^{\frac{1}{2}}$. In these cases the tangential velocity is a potential vortex and $W(\zeta = 0)$ is negative for $\eta > \eta_1/3^{\frac{1}{2}}$ and positive for $\eta < \eta_1/3^{\frac{1}{2}}$ (if this occurs for $\eta > 1$).

The second situation occurs when the above inequalities are reversed and V changes sign in the region $1 < \eta < \eta_1$. This will always occur if η_1 is sufficiently large. Now V will tend to zero at some value $\eta_2 > 1$. However, it will take an infinite time for the front to reach this value since the integral defining τ_{F} will diverge. In the region behind the front $W(\zeta = 0)$ is always negative, feeding fluid into the Ekman layer.

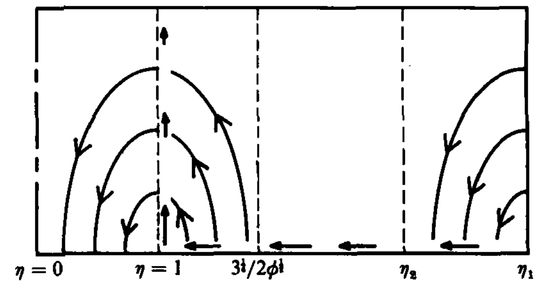


FIGURE 3. Sketch of the flow in a rotating cylinder with uniform outflow at the drain. Bottom friction is included and uniform inflow through a cylindrical boundary where constant tangential velocity is imposed.

While the front is approaching its asymptotic value the flow ahead of the front is tending to *its* asymptotic state in which all the radial flow is in the Ekman layer (which therefore comes from behind the front). The resulting flow is sketched in figure 3. When $\eta_1 \to \infty$, η_2 does also and the flow tends to the asymptotic state discussed previously with the Ekman-layer flow coming from infinity.

While the outer part of the flow tends to a steady state, the part above the drain hole does not (without internal viscosity), as can be seen from (5.14). Numerical analysis of (5.15) is simplified because the integral depends only on ξ (for fixed ϕ) and will therefore give

$$\eta U_\theta = \xi^2 = F(\eta^2 \mathrm{e}^\tau, \phi). \tag{5.35}$$

The asymptotes of this function are

$$\eta U_\theta = 1 + \ln(\eta^2 \mathrm{e}^\tau) + \phi[\ln(\eta^2 \mathrm{e}^\tau)]^2 + \ldots, \tag{5.36}$$

for small $\ln(\eta^2 \mathrm{e}^\tau)$. For $\ln(\eta^2 \mathrm{e}^\tau) \to \infty$,

$$\begin{aligned} \eta U_\theta &\to \frac{3^{\frac{3}{2}}}{4\phi}, \quad \phi < \tfrac{1}{4}3^{\frac{1}{2}}; \\ \eta U_\theta &\to \left[\frac{1}{4\phi} + \left(1 + \left(\frac{1}{4\phi}\right)^2\right)^{\frac{1}{2}}\right]^2, \quad \phi > \tfrac{1}{4}\sqrt{3}. \end{aligned} \tag{5.37}$$

These limits are supplemented by some numerical work and presented in figures 4 and 5 for $\phi = 0.1$. It is seen that as $\tau \to \infty$ the solution tends to a steady potential vortex except in a core which becomes ever thinner with time. This should be compared with the result for zero wall friction ($\phi = 0$) in which the circulation continues to increase with time. This limit is also indicated on figure 4.

With the Einstein–Li boundary condition the flow above the drain hole is unchanged from the above if $\eta_1 > 3^{\frac{3}{2}}/2\phi^{\frac{1}{2}}$ (or $\eta_1 > (4\phi)^{-1} + (1 + (4\phi)^{-2})^{\frac{1}{2}}$ if $\phi > \frac{1}{4}3^{\frac{1}{2}}$. However, if these inequalities are not satisfied it will be modified since the front of the incoming fluid will pass through $\eta = 1$ in the finite time τ_{F}. Therefore when $\tau > \tau_{\mathrm{F}}$ the solution will be

$$\eta U_\theta = \eta_1^2 \quad \text{for } \eta_{\mathrm{F}} < \eta < 1, \tag{5.38}$$

and will be the same as in figure 5 for $\eta < \eta_{\mathrm{F}}(\tau)$. Since the circulation in (5.38) is smaller than the asymptotic values given by (5.37), $\eta_{\mathrm{F}}(\tau)$ occurs at the intersection of η_1^2 with the time-dependent inner core. The nature of the solution is still a potential vortex outside of a shrinking core.

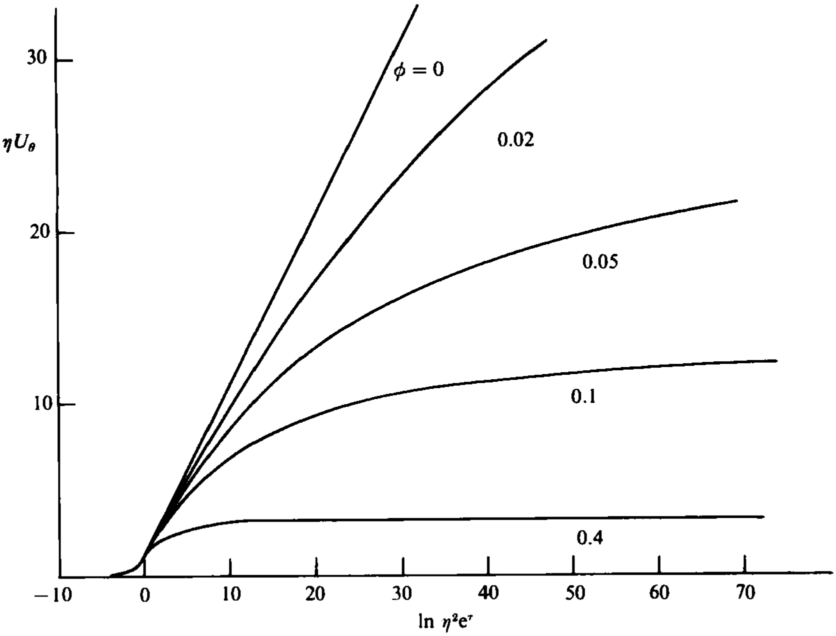


FIGURE 4. Circulation distribution in a rotating cylinder with uniform outflow at the drain. Bottom friction is included but there is no internal viscosity.

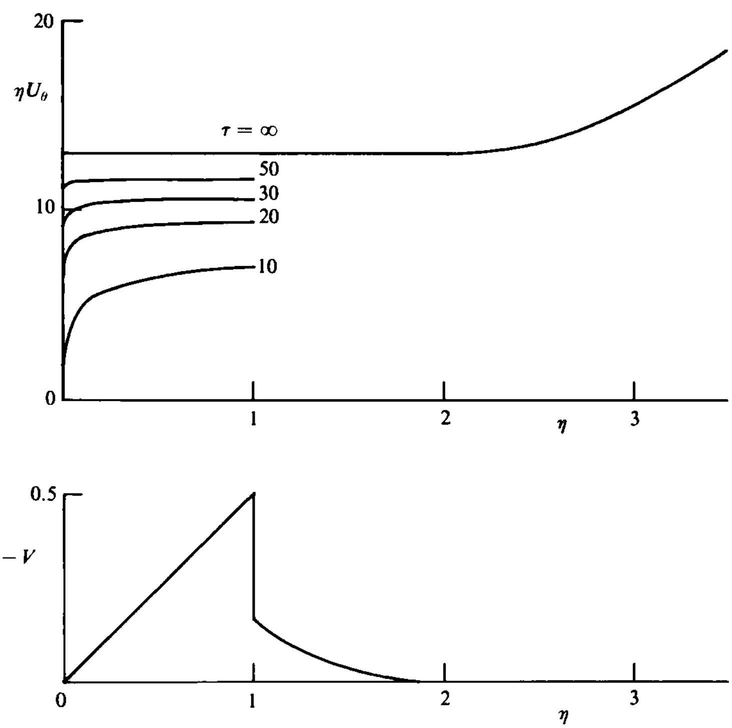


FIGURE 5. Circulation distribution and radial velocity in a rotating cylinder with uniform outflow at the drain. There is bottom friction but not internal viscosity. This is plotted for $\phi = 0.1$ with τ as a parameter. ϕ is defined in §5. Some of the circulation curves have been omitted for $\eta > 1$.

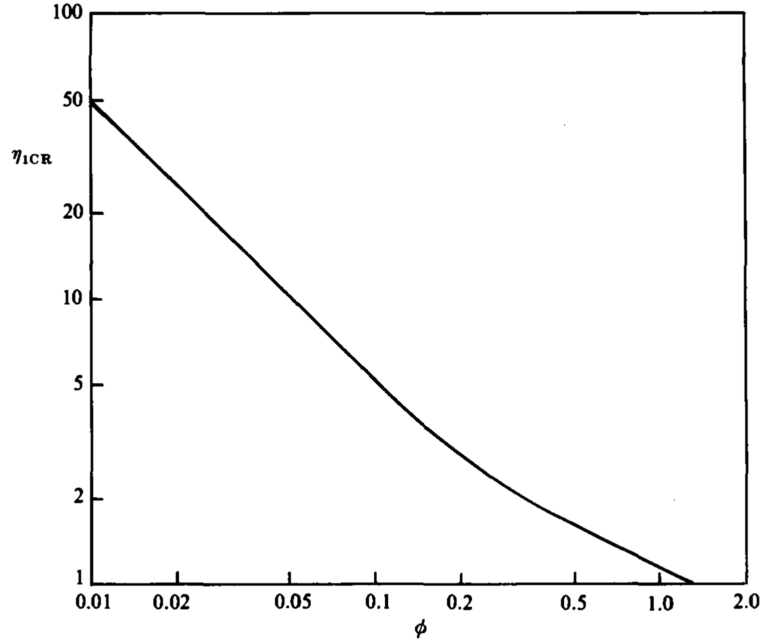


FIGURE 6. Critical radius of cylinder versus ϕ for uniform outflow from rotating cylinder with uniform inflow from cylindrical boundary. Bottom friction and internal viscosity are both included. If the radius of the cylinder is greater than the critical value the circulation about the viscous core is determined by the bottom friction. If the radius is smaller than this value the circulation is the same as the value imposed at the outer radius of the cylinder.

Ultimately, the interior viscosity will become important in these shrinking cores. Because the circulation is constant in all the cases above, the flows will tend to steady states which are Burgers' vortex, as in (4.28), but with different values of the circulation. The solution is

$$\eta U_\theta = \Gamma(1-\epsilon^{-\eta^2/4N}), \tag{5.39}$$

for $N = \nu H/u_\infty R^2 \ll 1$ and Γ given by (5.37) or by (5.38) depending on the parameters ϕ and η_1. The results are:

$$\text{(i)} \qquad \phi < \tfrac{1}{4}3^{\frac{1}{2}}; \quad \Gamma = \eta_1^2, \quad \eta_1 < \frac{3^{\frac{3}{4}}}{2\phi^{\frac{1}{2}}}; \quad \Gamma = \frac{3^{\frac{3}{2}}}{4\phi}, \quad \eta_1 > \frac{3^{\frac{3}{4}}}{2\phi^{\frac{1}{2}}}; \tag{5.40}$$

$$\text{(ii)} \qquad \left.\begin{array}{c} \phi > \tfrac{1}{4}3^{\frac{1}{2}}; \quad \Gamma = \eta_1^2, \quad \eta_1 < (4\phi)^{-1} + (1+(4\phi)^{-2})^{\frac{1}{2}}; \\ \Gamma = [(4\phi)^{-1} + (1+(4\phi)^{-2})^{\frac{1}{2}}]^2, \quad \eta_1 > (4\phi)^{-1} + (1+(4\phi)^{-2})^{\frac{1}{2}}. \end{array}\right\} \tag{5.41}$$

Essentially, this means that if η_1 is larger than a critical value which depends on ϕ the circulation is determined by boundary friction as if the outer wall were at infinity, while if η_1 is smaller than this value it is determined by the imposed circulation at the outer boundary and is the same as if there were no wall friction. The critical value is plotted in figure 6. In the Einstein–Li experiments $\phi = O(10^{-3})$ while η_1 is only about 17, so the wall friction does not affect the circulation in this case. It should be possible to devise similar experiments in which the wall friction is the dominant effect.

The limited free-surface analysis described at the end of §4 is also applicable here with K redefined by $K = \Gamma^2/NG$.

5.2. *The effect of viscosity at both upper and lower boundaries*

The slow-efflux problem with uniform suction will be considered again with friction at the upper surface also included. In this case there will be a steady-state solution, even without internal viscosity. In the interest of brevity only the steady state will be discussed here.

The boundary conditions are

$$\left.\begin{aligned} W(\zeta = 1) &= -\phi \frac{1}{\eta}\frac{\partial}{\partial\eta}\frac{\eta(\eta U_\theta - \eta^2)}{(\nu U_\theta)^{\frac{1}{2}}}, \quad \text{all } \eta; \\ W(\zeta = 0) &= -1, \quad \eta < 1; \\ &= \phi \frac{1}{\eta}\frac{\partial}{\partial\eta}\frac{\eta(\eta U_\theta - \eta^2)}{(\eta U_\theta)^{\frac{1}{2}}}, \quad \eta > 1. \end{aligned}\right\} \tag{5.42}$$

Therefore V is determined from

$$\frac{1}{\eta}\frac{\partial}{\partial\eta} V = W(\xi = 0) - W(\zeta = 1); \tag{5.43}$$

and is found to be

$$V = -\tfrac{1}{2}\eta + \phi\frac{\eta U_\theta - \eta^2}{(\eta U_\theta)^{\frac{1}{2}}}, \quad \eta < 1; \tag{5.44}$$

$$V = -\frac{1}{2}\frac{1}{\eta} + 2\phi\frac{\eta U_\theta - \eta^2}{(\eta U_\theta)^{\frac{1}{2}}}, \quad \eta > 1. \tag{5.45}$$

In order to have a steady-state solution either $V = 0$ or ηU_θ is constant as discussed in the last section.

For the region $\eta > 1$ the result for ηU_θ is the same as in the last section but with ϕ in (5.25) and (5.26) replaced by 2ϕ. Thus,

$$\eta U_\theta = \frac{3^{\frac{3}{2}}}{8\phi}, \quad \eta < \frac{3^{\frac{1}{4}}}{2^{\frac{3}{2}}\phi^{\frac{1}{2}}}; \tag{5.46}$$

$$\eta U_\theta = [(8\phi\eta)^{-1} + (\eta^2 + (8\phi\eta)^{-2})^{\frac{1}{2}}]^2, \quad \eta > \frac{3^{\frac{1}{4}}}{2^{\frac{3}{2}}\phi^{\frac{1}{2}}}. \tag{5.47}$$

If $\phi > \frac{1}{8}3^{\frac{1}{2}}$ only (5.47) occurs.

The radial velocity is given by

$$V = -\frac{1}{2}\frac{1}{\eta} + 2\phi\left(\frac{3^{\frac{3}{2}}}{8\phi} - \eta^2\right)\Big/\left(\frac{3^{\frac{3}{2}}}{8\phi}\right)^{\frac{1}{2}}, \quad \eta < \frac{3^{\frac{1}{4}}}{2^{\frac{3}{2}}\phi^{\frac{1}{2}}}; \tag{5.48}$$

$$= 0, \quad \eta > \frac{3^{\frac{1}{4}}}{2^{\frac{3}{2}}\phi^{\frac{1}{2}}}. \tag{5.49}$$

In the region where $V = 0$ the radial flux from infinity is carried equally in the two Ekman layers and $W = 0$ in this region. In the inner region described by (5.46) (when it occurs) the vertical velocity varies linearly between

$$W(\zeta = 0) = 2^{\frac{3}{2}}3^{\frac{1}{4}}\phi^{\frac{3}{2}}\left(\frac{3^{\frac{1}{2}}}{8\phi} - \eta^2\right)\Big/\eta, \tag{5.50}$$

and

$$W(\zeta = 1) = - W(\zeta = 0). \tag{5.51}$$

Thus part of the Ekman flux flows out of both Ekman layers before the vertical shear layer is reached. The part remaining in the lower layer enters the vertical shear layer.

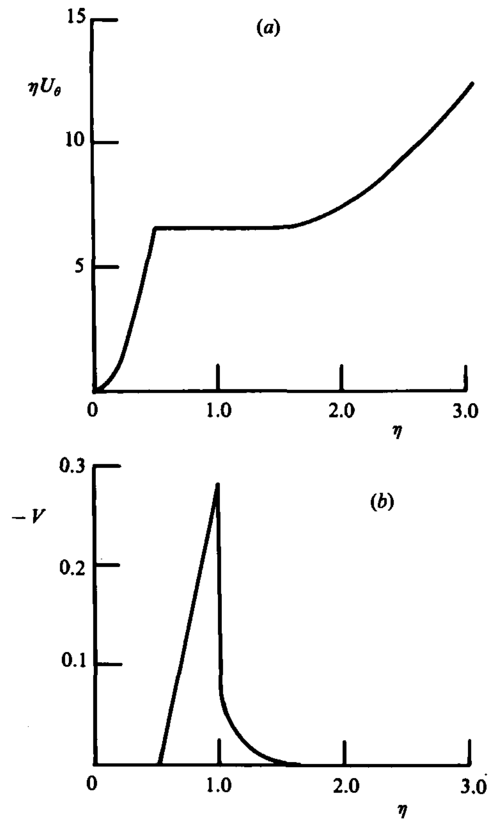


FIGURE 7. Steady-circulation distribution and radial velocity in a rotating cylinder with uniform outflow at the drain. Bottom and top friction are included but no internal viscosity. This is plotted for $\phi = 0.1$. Compare with figure 5.

For $\eta < 1$, the $V = 0$ case from (5.44) gives

$$\eta U_\theta = \eta^2 [(4\phi)^{-1} + (1+(4\phi)^{-2})^{\frac{1}{2}}]^2. \tag{5.52}$$

This is solid-body rotation. This must be joined to the region $\eta > 1$ by sections where $V \neq 0$ but $\eta U_\theta =$ constant. Thus for $\phi > \frac{1}{8}3^{\frac{1}{2}}$ one finds

$$\eta U_\theta = \eta^2 [(4\phi)^{-1} + (1+(4\phi)^{-2})^{\frac{1}{2}}]^2, \quad \eta < \frac{(8\phi)^{-1} + (1+(8\phi)^{-2})^{\frac{1}{2}}}{(4\phi)^{-1} + (1+(4\phi)^{-2})^{\frac{1}{2}}}; \tag{5.53}$$

$$\eta U_\theta = [(8\phi)^{-1} + (1+(8\phi)^{-2})^{\frac{1}{2}}]^2, \quad \frac{(8\phi)^{-1} + (1+(8\phi)^{-2})^{\frac{1}{2}}}{(4\phi)^{-1} + (1+(4\phi)^{-2})^{\frac{1}{2}}} < \eta < 1. \tag{5.54}$$

For $\phi < \frac{1}{8}3^{\frac{1}{2}}$ the solution is

$$\eta U_\theta = \eta^2 [(4\phi)^{-1} + (1+(4\phi)^{-2})^{\frac{1}{2}}]^2, \quad \eta < \frac{3^{\frac{3}{4}}/2^{\frac{3}{2}}\phi^{\frac{1}{2}}}{(4\phi)^{-1} + (1+(4\phi)^{-2})^{\frac{1}{2}}}; \tag{5.55}$$

$$\eta U_\theta = \frac{3^{\frac{3}{2}}}{8\phi}, \quad \frac{3^{\frac{3}{4}}/2^{\frac{3}{2}}\phi^{\frac{1}{2}}}{(4\phi)^{-1} + (1+(4\phi)^{-2})^{\frac{1}{2}}} < \eta < 1. \tag{5.56}$$

This result if plotted in figure 7 for $\phi = 0.1$ together with V which is negative in the regions where ηU_θ is constant and tends to zero at the edges of this region. (This figure

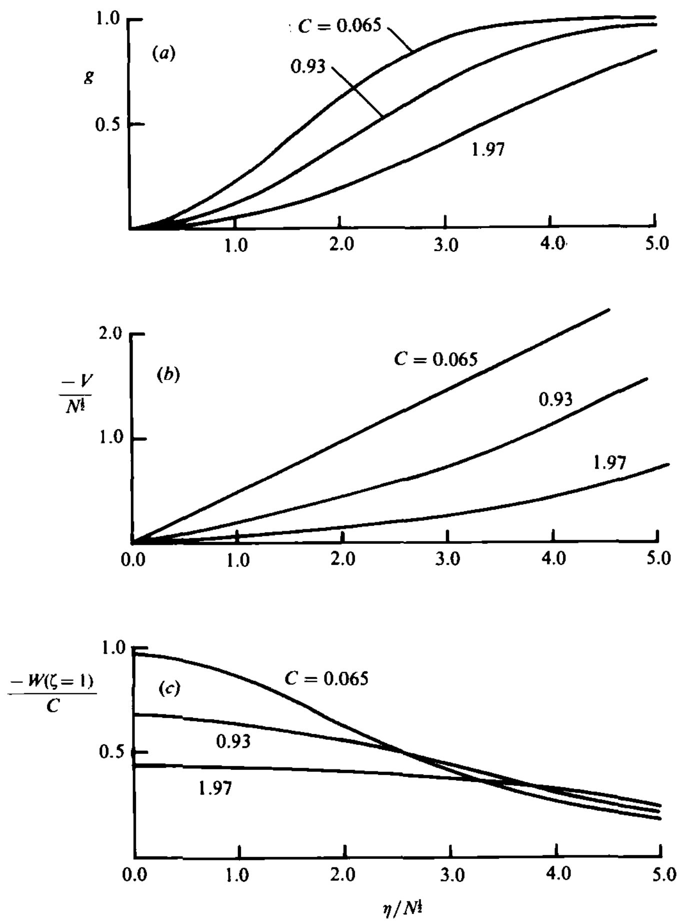


FIGURE 8. The effect of internal viscosity on the central core of the flows described in figure 7 (g defined by (5.52), C by (5.59)) and in figure 10 (g defined by (6.18), C by (6.20)).

should be compared with figure 5 where there is only bottom friction.) The flow spirals straight down from the upper Ekman layer to the drain hole in the inner region where $V = 0$, while there is a radially inward component in the remaining part with some of the flow being out of the upper Ekman layer and some out of the vertical shear layer.

Internal viscosity becomes important in the central region when the width of the region, which is of order $\phi^{\frac{1}{2}}$ for small ϕ, becomes comparable to the viscous length $N^{\frac{1}{2}}$. The problem is to solve (4.1) for the steady state, when V is given by (5.44). One can make use of the fact that ϕ must be small (since N is small) by noting that ηU_θ must match into $3^{\frac{3}{2}}/8\phi$ when η is large compared to $\phi^{\frac{1}{2}}$. This suggests a change of variables to

$$s = \frac{\eta}{N^{\frac{1}{2}}}, \quad g = \frac{8\phi}{3^{\frac{3}{2}}}\,\eta U_\theta. \tag{5.57}$$

In these variables

$$V = -\tfrac{1}{2}N^{\frac{1}{2}}s + \frac{3^{\frac{3}{4}}}{2^{\frac{3}{2}}}\,\phi^{\frac{1}{2}}g^{\frac{1}{2}} + O(\phi^{\frac{3}{2}}N). \tag{5.58}$$

Therefore as $\phi \to 0$, with $\phi/N = O(1)$, g must satisfy

$$\left.\begin{aligned} [-\tfrac{1}{2}s + Cg^{\frac{1}{2}}]\frac{\mathrm{d}g}{\mathrm{d}s} &= \frac{\mathrm{d}^2 g}{\mathrm{d}s^2} - \frac{1}{s}\frac{\mathrm{d}g}{\mathrm{d}s}, \\ C &= \frac{3^{\frac{3}{4}}}{2^{\frac{3}{2}}}\frac{\phi^{\frac{1}{2}}}{N^{\frac{1}{2}}}. \end{aligned}\right\} \tag{5.59}$$

This is to be solved with boundary conditions $g = 0$ at $s = 0$ and $g \to 1$ as $s \to \infty$. The parameter C defined above is independent of the drain velocity since

$$\frac{\phi}{N} = 0.67\left(\frac{\Omega R^2}{\nu}\right)^{\frac{1}{2}}\frac{R}{H}, \tag{5.60}$$

and is likely to be of order one on laboratory experiments with water. Equation (5.59) has been solved numerically for several values of C. The results are presented in figure 8. When C is small the solution approaches Burgers' vortex.

6. Turner's problem: application to the HBG experiment

Turner (1966) considered a laboratory model for a tornado which is similar to the drain problems. In the present geometry this may be described as a flow generated in a rotating cylinder by an imposed vertical velocity profile at the bottom consisting of a central suction surrounded by an annular blowing region with the same volume flow rate so there is no net fluid withdrawn from the vessel. Viscous effects are allowed at the upper boundary and in the interior but not at the lower boundary. In a tornado the prescribed velocity profile is in the clouds while the 'upper' boundary is the ground. The problem described seems to model the HBG vortices. The imposed velocity profile is embedded in the layer of turbulence produced by the oscillating grid and is not known in detail. The analogy of these vortices with tornados was pointed out by Hopfinger & Browand (1982).

The vertical profile will be taken as

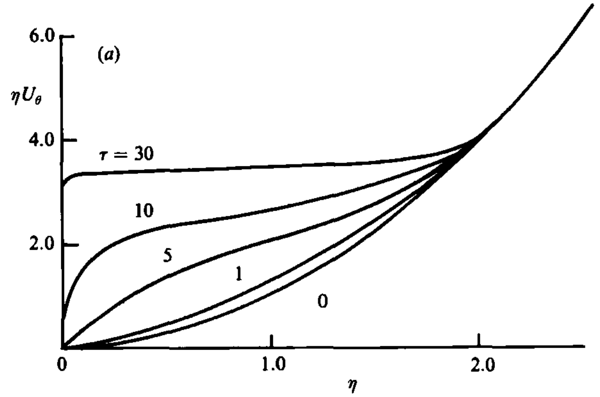
$$\left.\begin{aligned} W(\zeta = 0) &= -\tfrac{1}{2}(2-\eta)(1-\eta), \quad \eta < 2; \\ &= 0, \quad \eta > 2. \end{aligned}\right\} \tag{6.1}$$

This is similar to the profiles assumed by Turner but somewhat simpler. Much of what is discussed here can be generalized to profiles which have the same general characteristics as (6.1). Some of the parameters used previously have to be redefined; R is now the radius where $W(\zeta = 0)$ changes sign; u_∞ is the magnitude of the velocity at $\eta = 0$.

For the purposes of flow visualization HBG operate their device without the upper plate. Therefore, it is appropriate to consider first the problem without friction at the upper boundary. Initially the transient case will be treated without internal viscosity.

The problem is the same as in §3 with W varying linearly between zero at the top surface and the value given by (6.1) at the bottom. The radial velocity is therefore given by

$$\left.\begin{aligned} V &= -\tfrac{1}{8}\eta(2-\eta)^2, \quad \eta < 2; \\ &= 0, \quad \eta > 2. \end{aligned}\right\} \tag{6.2}$$



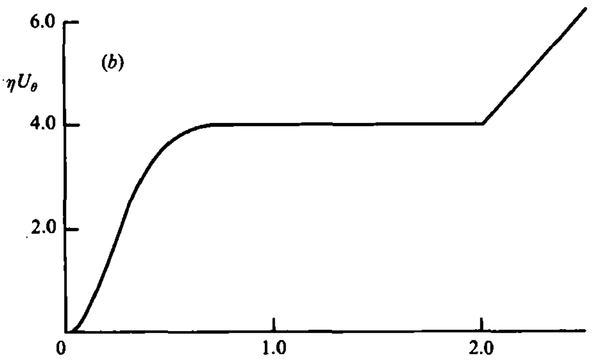


FIGURE 9. Circulation distribution in Turner's problem (§6) without friction at the upper surface; (a) without internal viscosity, with τ as a parameter; (b) steady-state solution with internal viscosity, $N = 0.025$. Inner core is the same as Burgers' vortex.

The integrations indicated in (3.21) can be carried out to obtain

$$-2\ln\left[\frac{\eta}{2-\eta}\right]-\frac{4}{2-\eta} = \tau-2\ln\left[\frac{\xi}{2-\xi}\right]-\frac{4}{2-\xi}, \quad \eta < 2; \tag{6.3}$$

$$\xi^2 = \eta^2, \quad \eta > 2; \tag{6.4}$$

where, as before, $\xi^2 = \eta U_\theta$. The flow is solid-body rotation for $\eta > 2$ since there is no radial velocity to disturb it. When $\eta < 2$ it is not possible to solve explicitly for ξ but this is easily done numerically with the result shown in figure 9(a). The asymptotes are clearly seen from (6.3). As $\tau \to \infty$ with η fixed (but not equal to zero) the only other term which can balance τ is the last one in (6.3). Therefore $\xi \to 2$. A few terms in an expansion gives

$$\eta U_\theta = \xi^2 = 4-\frac{16}{\tau^2}+\frac{16}{\tau^2}\left[1-2\ln\left(\frac{\tau}{2}\right)\right]+\dots, \quad \tau \to \infty. \tag{6.5}$$

For small η and ξ several other terms balance, giving

$$\eta U_\theta = \xi^2 = \eta^2 e^\tau, \quad \eta^2 e^\tau \ll 1. \tag{6.6}$$

This flow does not settle down to a steady state as $\tau \to \infty$; the central core continues to shrink. However, unlike the flow in §3 the circulation about the core region tends to a finite value as $\tau \to \infty$. Physically, this is because parcels of rotating fluid do not move very far radially in this problem. The flow more closely resembles that of §5.2 where the upflow was confined to a vertical shear layer.

The effect of internal viscosity is easy to include. The flow tends to Burgers' vortex as $\tau \to \infty$ with

$$\eta U_\theta = 4(1-e^{-\eta^2/4N}), \quad \eta < 2; \tag{6.7}$$

$$\eta U_\theta = \eta^2, \quad \eta > 2; \tag{6.8}$$

where $N = \nu H/U_\infty R^2$ is small. This result is independent of the details of the specifying vertical profile; the flow is Burgers' vortex with circulation the same as in the solid-body rotation at the edge of the structure (which is at $\eta = 2$ in the above solution). This is plotted in figure 9(*b*) for $N = 0.025$. From this figure it is clear that viscosity is also important near the corner in the profile at $\eta = 2$. This is a discontinuity in vorticity which will be smoothed out by viscosity but not in a steady manner. An unsteady viscous layer will develop in which the irrotational inner region encroaches into the constant-vorticity outer region. As this proceeds the circulation about the inner vortex core will slowly increase with time. This unsteady layer is not required if there is any friction at the upper surface, as can be seen from the analysis below.

By observing the depth and shape of the dimple which vortices induce in the free surface HBG have estimated the maximum vorticity in the vortex. For one set of observations they find $\omega_{max} = 80\Omega$. From (6.8) the vorticity is

$$\frac{\omega_z}{\Omega} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\eta U_\theta = \frac{2}{N}e^{-\eta^2/4N}, \quad \eta < 2. \tag{6.9}$$

This gives $\omega_{max} = 2\Omega/N$ from which an estimate of N can be obtained; $N = 0.025$. From the definition of N one can estimate u_∞, the maximum suction velocity. Taking $\nu = 0.01$, $H = 30$ cm and $R = 1$ cm (the outer edge of the vortex is at a radius of about 2 cm) gives $u_\infty \approx 12$ cm/s which can be compared to the maximum velocity of the oscillating grid of about 100 cm/s.

The effect of surface friction at the upper boundary will now be considered. Interior viscosity will be neglected initially and attention will be directed to the steady-state solution which exists for this case. The method to be used is the same as in §5.2.

The boundary velocities are

$$W(\zeta = 1) = -\phi\frac{1}{\eta}\frac{\partial}{\partial \eta}\frac{\eta(\eta U_\theta - \eta^2)}{(\eta U_\theta)^{\frac{1}{2}}}, \tag{6.10}$$

and $W(\zeta = 0)$ is given by (6.1). Since W varies linearly between these values, V is determined from

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\eta V = W(\zeta = 0) - W(\zeta = 1),$$

and is found to be

$$V = -\tfrac{1}{8}\eta(2-\eta)^2 + \phi\frac{(\eta U_\theta - \eta^2)}{(\nu U_\theta)^{\frac{1}{2}}}, \quad \eta < 2; \tag{6.11}$$

$$= \phi\frac{(\eta U_\theta - \eta^2)}{(\eta U_\theta)^{\frac{1}{2}}}, \quad \eta > 2. \tag{6.12}$$

For a steady-state solution either V is zero or ηU_θ is constant. For the part of the solution where V is zero, (6.11) and (6.12) give

$$\eta U_\theta = \left[\frac{\eta(2-\eta)^2}{16\phi} + \left(\eta^2 + \left[\frac{\eta(2-\eta)^2}{16\phi}\right]^2\right)^{\frac{1}{2}}\right]^2, \quad \eta < 2; \tag{6.13}$$

$$\eta U_\theta = \eta^2, \quad \eta > 2. \tag{6.14}$$

This function is either monotonically increasing (Turner restricted it to this case) or contains a maximum and a minimum value for $\eta < 2$. In the latter case the complete solution must have a region of constant ηU_θ inserted from the minimum in such a way as to truncate the maximum as in §5.2. A little analysis shows that the relative maxima are at the roots of

$$-(2-\eta)^2(2-3\eta)(2+\eta) = (16\phi)^2. \tag{6.15}$$

There are two roots if

$$\phi < \left[\frac{2\sqrt{3}/3-1}{12}\right]^{\frac{1}{2}} = 0.114, \tag{6.16}$$

and none otherwise. The resulting velocity is plotted for several values of ϕ in figure 10. As $\phi \to 0$ the position of the minimum approaches $\eta = 2$ and when both η and ϕ are small (6.14) gives a region of solid-body rotation. The solution thus becomes (as $\phi \to 0$);

$$\left.\begin{aligned} &\eta U_\theta \approx \eta^2/4\phi^2, \quad \eta < 4\phi; \\ &\eta U_\theta = 4, \quad 4\phi < \eta < 2; \\ &\eta U_\theta = \eta^2, \quad \eta > 4. \end{aligned}\right\} \tag{6.17}$$

The radial velocity is determined from (6.11) and is non-zero in the annular regions where ηU_θ is constant. This is also plotted in figure 10.

The vertical component of velocity is equal to $W(\zeta = 0)$ in the regions where $V = 0$; the flow spirals straight up in an outer annular region into the Ekman layer and straight down out of the Ekman layer in an inner core. If $\phi > 0.114$ the whole flow consists of only these two regions. If $\phi < 0.114$ there is an intermediate annular region with $V < 0$, where some of the upflow turns back before reaching the top. If ϕ is very small this region dominates and in the limit all of the upflow turns back before reaching the upper boundary; $W(\zeta = 1) = 0$. This limit is quite non-uniform. For small but finite ϕ, $W(\zeta = 1)$ is of order ϕ over most of the region but is near 1 in a thin central jet with width of order ϕ which erupts out of the Ekman layer. The jet region coincides with the central region of solid-body rotation.

The effect of interior viscosity can now be assessed. Since N is assumed to be small it is clear that viscosity is unimportant unless ϕ is small. Referring to the limiting solution given by (6.17) it is seen that the radius of the inner core is of order ϕ. Therefore viscosity will become important when $N^{\frac{1}{2}} = O(\phi)$, differing from the results of §5.2. The appropriate scaling of (4.1) is now

$$s = \frac{\eta}{N^{\frac{1}{2}}}, \quad g = \tfrac{1}{4}\eta U_\theta, \tag{6.18}$$

and from (6.11)

$$V = -\tfrac{1}{2}N^{\frac{1}{2}}s + 2\phi g^{\frac{1}{2}} + O(N). \tag{6.19}$$

As $\phi \to 0$, g must satisfy

$$\left.\begin{aligned} (-\tfrac{1}{2}s + Cg^{\frac{1}{2}})\frac{dg}{ds} &= \frac{d^2g}{ds^2} - \frac{1}{s}\frac{dg}{ds} \\ C &= 2\phi/N^{\frac{1}{2}}. \end{aligned}\right\} \tag{6.20}$$

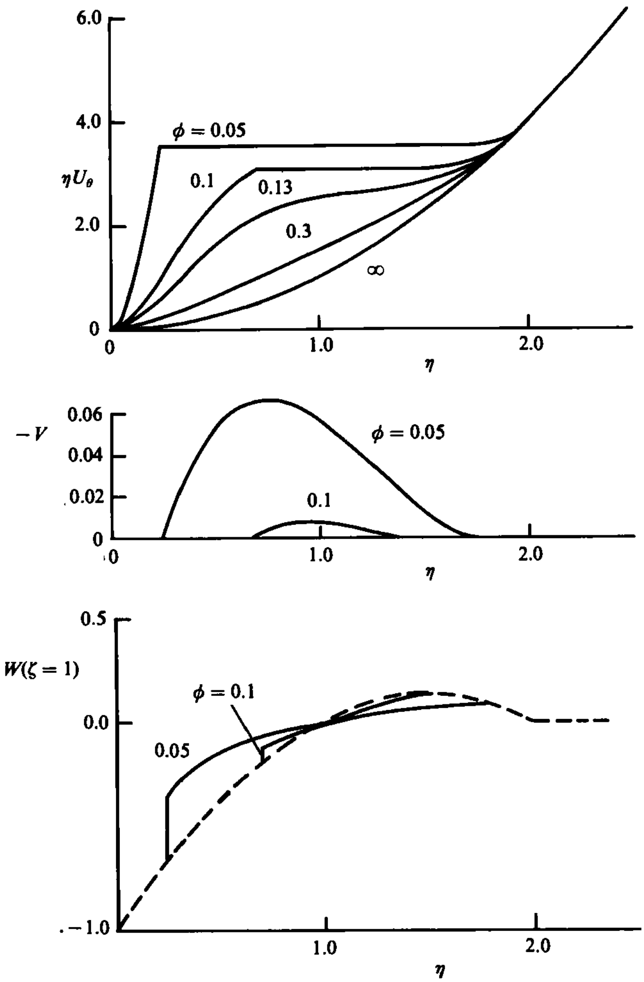


FIGURE 10. Steady-circulation distribution, radial velocity and vertical velocity in Turner's problem with friction at upper surface but without internal viscosity. This is plotted with various values of ϕ as parameter.

This is to be solved with boundary conditions $g = 0$ at $s = 0$ and $g \to 1$ as $s \to \infty$. This is exactly the same problem as described by (5.59) except that g is scaled differently and C has a different definition. (Note that with this definition C is independent of the viscosity.) Figure 8 is still appropriate. As $C \to 0$ the solution tends to Burgers' vortex.

The parameters in the HBG experiment when the upper cover is in place may be estimated. Assuming that the estimate $u_\infty = 12$ cm/s is still valid, and using $H = 52$ cm, $R = 1$ cm, $\nu = 0.01$ cm²/s:

$$\left.\begin{aligned} N &= \frac{\nu H}{u_\infty R^2} = 0.043; \\ \phi &= \frac{0.67(\nu\Omega)^{\frac{1}{2}}}{u_\infty} = 0.014; \\ C &= \frac{2\phi}{N^{\frac{1}{2}}} = 0.14. \end{aligned}\right\} \tag{6.21}$$

Since C is so small, friction at the upper boundary is not very important in this flow.

Turner's analysis of this problem differs from that presented above. While he correctly assessed the physics of the problem and introduced the use of the nonlinear Ekman suction condition, he made an unnecessary assumption about the nature of the vertical velocity at the edge of the Ekman layer. He assumed that $W(\zeta = 1) = \frac{1}{2}W(\zeta = 0)$ which is not true. His results for ηU_θ seem to be insensitive to this assumption for certain ranges of the parameters.

7. Discussion and conclusion

The problems discussed in this paper all require that the Rossby number be small in order for quasi-two-dimensional flow structures to exist. One might enquire how small this parameter must be. Inspection of (1.9) suggests the condition $\epsilon^2(H/R)^2 \ll 1$, which is rather stringent because H/R is often large in experiments. From the definition of ϵ this means

$$\frac{u_\infty^2}{R^2\Omega^2} \ll 1. \tag{7.1}$$

This appears to be correct when u_θ is of order $R\Omega$ (by which it is scaled). However, this parameter is not small in either the Einstein–Li or the HBG experiments, but these flows are clearly quasi-two-dimensional. It seems that, in flows where u_θ becomes locally large, u_θ should be scaled with $R\Omega_{\max}$ (where $\Omega_{\max}$ is the maximum angular velocity on the axis) in order to assess the relative sizes of terms. This gives the condition

$$\frac{u_\infty^2}{R^2\Omega_{\max}^2} \ll 1. \tag{7.2}$$

For the HBG experiment the estimates

$$u_\infty = 12 \text{ cm/s}, \quad R = 1 \text{ cm}, \quad \Omega_{\max} = 40 \times 2\pi \text{ rad/s}$$

from §6 give

$$\frac{u_\infty}{R\Omega_{\max}} \approx 0.05.$$

In the Einstein–Li experiment estimates of the parameters have been made from their data. For the case with smaller flow rate ($u_\infty = 22$ cm/s, $R = 0.6$ cm, $\Omega_{\max} = 1500\Omega$, $\Omega = 0.8$ rad/s)

$$\frac{u_\infty}{R\Omega_{\max}} = 0.03,$$

while for the case with high flow rate ($u_\infty = 66$ cm/s, $R = 0.6$ cm, $\Omega_{\max} \approx 750\Omega$, $\Omega = 0.5$ rad/s)

$$\frac{u_\infty}{R\Omega_{\max}} = 0.3.$$

Therefore these flows satisfy the modified condition expressed by (7.2).

It is believed that this study gives some insight into a class of nonlinear rotating flows which have a structure similar to those of well-understood linear problems in which an inviscid interior flow is controlled by Ekman boundary layers. These rotating drain problems have very nonlinear interior flows despite the limitation of small Rossby number (as measured by $u_\infty/R\Omega_{\max}$). Some simple experiments are planned to verify the structure of these flows.

Appendix A. Solution of the linearized problem

The problem posed in §2 is to solve

$$\frac{\partial^2}{\partial r^2}\left[\frac{\partial^2 U_{z0}}{\partial \zeta^2}+\left[\frac{H}{R}\right]^2\frac{1}{\eta}\frac{\partial}{\partial \eta}\frac{\partial \eta U_{z0}}{\partial \eta}\right]+4\frac{\partial^2 U_{z0}}{\partial \zeta^2}=0 \tag{A 1}$$

with boundary and initial conditions

$$\left.\begin{aligned} U_{z0} &= 0 \quad \text{at } \zeta = 1,\\ U_{z0} &= W_{\mathrm{B}}(\eta)\,F(T/T_{\mathrm{s}}) \quad \text{at } \zeta = 0,\\ U_{z0} &= 0 \quad \text{at } T = 0.\end{aligned}\right\} \tag{A 2}$$

The solution may be found in the form of a Bessel integral,

$$U_{z0}(\eta,\zeta,T)=\int_0^\infty U(\lambda,\zeta,T)\,J_0(\lambda\eta)\,\lambda\,\mathrm{d}\lambda, \tag{A 3}$$

in which U is the solution of

$$\frac{\partial^2}{\partial T^2}\left[\frac{\partial^2 U}{\partial \zeta^2}-\left[\frac{H}{R}\right]^2\lambda^2 U\right]+4\frac{\partial^2 U}{\partial \zeta^2}=0, \tag{A 4}$$

$$U = 0 \quad \text{at } \zeta = 1,$$

$$U = U_{\mathrm{B}}(\lambda)\,F(T/T_{\mathrm{s}}) \quad \text{at } \zeta = 0,$$

$$U = 0 \quad \text{at } T = 0,$$

where

$$U_{\mathrm{B}}=\int_0^R W_{\mathrm{B}}(\eta)\,J_0(\eta\lambda)\,\eta\,\mathrm{d}\eta. \tag{A 5}$$

The non-homogeneous term may be shifted to the partial differential equation by the change of variable

$$U = U_{\mathrm{B}}\,F(T/T_{\mathrm{s}})\,(1-\zeta)+\tilde{U}, \tag{A 6}$$

for then $\tilde{U}$ must satisfy

$$\frac{\partial^2}{\partial T^2}\left[\frac{\partial^2 \tilde{U}}{\partial \zeta^2}-\tilde{\lambda}^2\tilde{U}\right]+4\frac{\partial^2 \tilde{U}}{\partial \zeta^2}=\tilde{\lambda}^2(1-\zeta)\,U_{\mathrm{B}}\frac{\mathrm{d}^2F}{\mathrm{d}T^2}, \tag{A 7}$$

where $\tilde{\lambda} = H\lambda/R$ and

$$\tilde{U}=0 \quad \text{at } \zeta = 1, \qquad \tilde{U}=0 \quad \text{at } \zeta=0, \qquad \tilde{U}=0 \quad \text{at } T=0.$$

The sine-series expansion

$$\tilde{U}=\sum_{n=1}^{\infty}\tilde{U}_n(T)\sin n\pi(1-\zeta), \tag{A 8}$$

$$1-\zeta=\sum_{n=1}^{\infty}a_n\sin n\pi(1-\zeta), \tag{A 9}$$

reduces the problem to an ordinary differential equation,

$$\frac{\mathrm{d}^2\tilde{U}_n}{\mathrm{d}T^2}+\frac{4\pi^2n^2\tilde{U}_n}{\tilde{\lambda}^2+\pi^2n^2}=-\frac{a_n\,U_{\mathrm{B}}\,\tilde{\lambda}^2}{\tilde{\lambda}^2+\pi^2n^2}\frac{\mathrm{d}^2F}{\mathrm{d}T^2}, \tag{A 10}$$

which may be solved by variation of parameters. The solution is

$$\tilde{U}_n = \left[-\frac{b_n}{\omega_n}\int_0^T \sin\omega_n T\,\frac{\mathrm{d}^2F}{\mathrm{d}T^2}\,\mathrm{d}T\right]\cos\omega_n T+\left[\frac{b_n}{\omega_n}\int_0^T \cos\omega_n T\,\frac{\mathrm{d}^2F}{\mathrm{d}T^2}\,\mathrm{d}T\right]\sin\omega_n T, \tag{A 11}$$

where

$$\omega_n = \frac{2\pi n}{(\lambda^2+\pi^2n^2)^{\frac{1}{2}}},$$

$$b_n = -\frac{a_n U_{\mathrm{B}}\tilde{\lambda}^2}{(\tilde{\lambda}^2+\pi^2N^2)}.$$

Using (A 3) the solution may be written as

$$U_{z0} = \int_0^\infty U_{\mathrm{B}} J_0(\lambda\eta)\,\lambda\,\mathrm{d}\lambda\,(1-\zeta)\,F(T/T_{\mathrm{s}})+\sum_{n=1}^{\infty}\int_0^\infty \tilde{U}_n(\lambda,T)\,J_0(\lambda\eta)\,\lambda\,\mathrm{d}\lambda\,\sin\eta\pi(1-\zeta). \tag{A 12}$$

Because of (A 5) and the Hankel inversion formula this may be written

$$U_{z0} = W_{\mathrm{B}}(\eta)\,(1-\zeta)\,F(T/T_{\mathrm{s}})+\sum_{n=1}^{\infty}\int_0^\infty \tilde{U}_n(\lambda,T)\,J_0(\lambda\eta)\,\lambda\,\mathrm{d}\lambda\,\sin n\pi(1-\zeta). \tag{A 13}$$

Appendix B. Effect of viscosity on accelerating swirl

The problem formulated in §4 is to solve

$$\left[\frac{\partial}{\partial\tau}-\tfrac{1}{2}s\frac{\partial}{\partial s}\right]\frac{\omega_z}{\Omega} = \frac{\omega_z}{\Omega}+\left[\frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}\right]\frac{\omega_z}{\Omega}, \tag{B 1}$$

with the asymptotic boundary condition

$$N\frac{\omega_z}{\Omega}\rightarrow\frac{2}{s^2} \quad \text{as } s\rightarrow\infty \tag{B 2}$$

and ω_z bounded at $s = 0$. It is assumed that initial conditions have decayed out and only a quasi-steady solution is sought. (The rate at which initial conditions damp out will be discussed later in this section.) One can readily see that there is no *steady* solution which satisfies the condition given by (4.7). In fact, it is clear that the circulation at large s increases with time. This can be seen from the inviscid solution or by the following reasoning. Define

$$A(s,\tau) = \int_0^s N\frac{\omega_z}{\Omega}\,s\,\mathrm{d}s. \tag{B 3}$$

Operating on (B 1) gives

$$\frac{\partial A}{\partial\tau} = \tfrac{1}{2}Ns^2\frac{\omega_z}{\Omega}+s\frac{\partial}{\partial s}N\frac{\omega_z}{\Omega}. \tag{B 4}$$

Using (B 2) shows

$$\frac{\partial A}{\partial\tau}\rightarrow 1 \quad \text{as } s\rightarrow\infty, \tag{B 5}$$

and therefore

$$A\rightarrow\tau+\text{some function of } s \quad \text{as } s\rightarrow\infty. \tag{B 6}$$

The above considerations suggest that a solution be sought in the form

$$N\frac{\omega_z}{\Omega} = \tau G_1(s)+G_2(s), \tag{B 7}$$

whence G_1 and G_2 must satisfy

$$-\tfrac{1}{2}s\,\frac{\mathrm{d}G_1}{\mathrm{d}s} = G_1 + \left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}}{\mathrm{d}s}\right]G_1, \tag{B 8}$$

$$G_1 - \tfrac{1}{2}s\,\frac{\mathrm{d}G_2}{\mathrm{d}s} = g_2 + \left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}}{\mathrm{d}s}\right]G_2. \tag{B 9}$$

The solutions are:

$$G_1 = \tfrac{1}{2}\mathrm{e}^{-\frac{1}{4}s^2}; \tag{B 10}$$

$$G_2 = \mathrm{e}^{-\frac{1}{4}s^2}\int_0^s \frac{\mathrm{e}^{\frac{1}{4}s^2}-1}{s}\,\mathrm{d}s + C_2\,\mathrm{e}^{-\frac{1}{4}s^2}. \tag{B 11}$$

The multiplicative constant in G_1 was chosen to satisfy (4.11) and G_2 automatically satisfies (4.7). The constant C_2 is a shift in the origin of τ and can be determined by matching ηU_θ calculated from

$$\eta U_\theta = \int_0^s N\frac{\omega_z}{\Omega}\,s\,\mathrm{d}s, \tag{B 12}$$

with the inviscid solution given by (3.23). Substituting (B 7), (B 10) and (B 11) into (B 12) the asymptotic behaviour may be determined to be:

$$\eta U_\theta = \ln(s^2) + \left[\tau + 2C_2 - \frac{1}{4}\int_0^\infty \ln(x)\,\mathrm{e}^{-\frac{1}{4}x}\,\mathrm{d}x\right] - \frac{4}{s^2} + \text{exponentially small terms}, \tag{B 13}$$

for large s. This must match with

$$\eta U_\theta = 1 + \tau + \ln(N) + \ln(s^2), \tag{B 14}$$

whence it is found that

$$\begin{aligned} C_2 &= \tfrac{1}{2}\ln(N) + \frac{1}{2} + \frac{1}{8}\int_0^\infty \ln(x)\,\mathrm{e}^{-\frac{1}{4}x}\,\mathrm{d}x \\ &= \tfrac{1}{2}\ln(N) + 0.9045. \end{aligned} \tag{B 15}$$

The solution is thus

$$N\frac{\omega_z}{\Omega} = (C_2 + \tfrac{1}{2}\tau)\,\mathrm{e}^{-\frac{1}{4}s^2} + \mathrm{e}^{-\frac{1}{4}s^2}\int_0^s \frac{\mathrm{e}^{\frac{1}{4}s^2}-1}{s}\,\mathrm{d}s. \tag{B 16}$$

While (B 16) gives an exact solution of the partial differential equation, it does not satisfy the exact initial conditions. It would be of value to know how fast the influence of the initial conditions decays away.

Suppose that a solution of the exact initial-value problem is known and suppose a neighbouring solution is also known with different initial conditions but with the same total amount of vorticity. This second solution can be regarded as the solution derived above. If G is the difference of these two solutions then it has the property that $\int_0^\infty Gs\,\mathrm{d}s = 0$. An estimate of the rate of decay of G is needed. A change of variables,

$$s_1 = \mathrm{e}^{\frac{1}{2}t}\,s, \quad G_1 = \mathrm{e}^{-\frac{1}{2}\tau}\,G, \quad \tau_1 = \mathrm{e}^{\tau} - 1, \tag{B 17}$$

reduces (B 1) to the two-dimensional heat equation,

$$\frac{\partial G_1}{\partial \tau_1} = \left[\frac{\partial^2}{\partial s_1^2} + \frac{1}{s_1}\,\frac{\partial}{\partial s_1}\right]G_1. \tag{B 18}$$

The solution of this may be found as an integral of the fundamental singular solution of the heat equation times the initial G. Upon expressing the result in terms of the original variables given in (B 17) and expanding for large τ one finds

$$G \rightarrow e^{-\tau} \quad \text{as } \tau \rightarrow \infty. \tag{B 19}$$

Therefore if $\tau = O(\ln N^{-1})$ one would find $G = O(N)$, which is small. The conclusion to be drawn is that the effect of the initial condition decays quite rapidly, becoming small at even moderate values of τ.

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